1. *Let* $\boldsymbol{r}_0$ *be the position vector of point* P_0 *with respect to some origin* O. *Give* **vector** *equations for the position vector* $\boldsymbol{r}$ *of any point* P *on*
(a) the line parallel to the vector $\boldsymbol{v}$ *that passes through* $\boldsymbol{r}_0$,
(b) the plane through $\boldsymbol{r}_0$ *that is parallel to both* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *(with* $\boldsymbol{u} \times \boldsymbol{v} \neq 0$*),*
(c) the plane through $\boldsymbol{r}_0$ *that is perpendicular to* $\boldsymbol{n}$,
(d) the plane through the three points $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ *(not all on a line).*
(e) If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *are 3 non co-planar vectors in a 3D vector space then any vector* $\boldsymbol{r}$ *can be expanded as* $\boldsymbol{r} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$. *Give vector equations for the components* α, β *and* γ *in terms of* $\boldsymbol{r}$ *and the basis vectors* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *(NOT necessarily orthogonal to each other).*

This elementary material was supposed to be a gift! See Math 222 and Math 234, e.g. Varberg, Purcell and Rigdon Chaps 13 (13.2, 13.4) and 14 (14.2, 14.4) and our own Weber & Arfken page 4 (eqn(1.3) and surrounding discussion about $\boldsymbol{r}$ being a special vector), pages 14-17, examples 1.2.1, 1.2.2, 1.3.1 and posted exercises 1.3.11, 1.4.8.

(a) $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$, where t is any real number.
(b) $\boldsymbol{r} = \boldsymbol{r}_0 + s\boldsymbol{u} + t\boldsymbol{v}$, where s, t are any real numbers.
(c) $(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0$.
(d) $\boldsymbol{r} = \boldsymbol{r}_0 + s(\boldsymbol{r}_1 - \boldsymbol{r}_0) + t(\boldsymbol{r}_2 - \boldsymbol{r}_0)$.
(e) $\alpha = [\boldsymbol{r} \cdot (\boldsymbol{b} \times \boldsymbol{c})]/[\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})]$, etc.

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2. *Consider the integral* $\oint_{\mathcal{C}} \boldsymbol{r} \times d\boldsymbol{r}$ *where* $\mathcal{C}$ *is a closed planar curve (i.e.* $\boldsymbol{r} \cdot \hat{\boldsymbol{z}} = 0$, $\forall \boldsymbol{r}$ *say).*
(a) Use your geometrical understanding of the cross-product to deduce the value of this integral. (Sketch plus a few words of explanation). Is the result a scalar or a vector?
(b) Rewrite the integral as a line integral (i.e. something of the form $\oint \boldsymbol{F} \cdot d\boldsymbol{r}$*) and evaluate that integral using Stokes (or Green's) theorem. USE CLEAR NOTATION!*
(c) If $\mathcal{C}$ *is the curve parametrized by* $\boldsymbol{r} = \hat{\boldsymbol{x}}\, a\cos t + \hat{\boldsymbol{y}}\, b\sin t$ *with* $\hat{\boldsymbol{x}}$ *and* $\hat{\boldsymbol{y}}$ *two mutually orthogonal unit vectors and* a, b *constants, express the integral* $\oint_{\mathcal{C}} \boldsymbol{r} \times d\boldsymbol{r}$ *as an integral over* t, *specifying the limits of integration. What is the specific value of the integral for this* $\mathcal{C}$*?*

This was supposed to be a gift! See posted exercise 1.11.1, your class notes on planetary motion and *EXAM 1, problem 2 (g)!*

(a) $\boldsymbol{r} \times d\boldsymbol{r} = 2\hat{\boldsymbol{z}}dA$ where dA is the area of the triangle $\boldsymbol{r}, d\boldsymbol{r}, \boldsymbol{r} + d\boldsymbol{r}$. So the integral is twice the area enclosed by $\mathcal{C}$ times $\hat{\boldsymbol{z}}$.
(b) $\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y \Rightarrow d\boldsymbol{r} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy$ and $\boldsymbol{r} \times d\boldsymbol{r} = \hat{\boldsymbol{z}}(xdy - ydx)$, then
$\oint_{\mathcal{C}} \boldsymbol{r} \times d\boldsymbol{r} = \hat{\boldsymbol{z}} \oint_{\mathcal{C}} (-y, x) \cdot d\boldsymbol{r} = \hat{\boldsymbol{z}} \int_A (\partial x/\partial x + \partial y/\partial y) dA = 2A\hat{\boldsymbol{z}}$, by Stokes (Green's) theorem.
(c)$\boldsymbol{r} = \hat{\boldsymbol{x}}\, a\cos t + \hat{\boldsymbol{y}}\, b\sin t \Rightarrow d\boldsymbol{r} = (-\hat{\boldsymbol{x}}\, a\sin t + \hat{\boldsymbol{y}}\, b\cos t)\, dt$ and $\boldsymbol{r} \times d\boldsymbol{r} = \hat{\boldsymbol{z}}ab\ dt$. Then $\oint_{\mathcal{C}} \boldsymbol{r} \times d\boldsymbol{r} = \int_0^{2\pi} \hat{\boldsymbol{z}}ab\ dt = 2\pi ab\hat{\boldsymbol{z}}$. This is twice the area of the ellipse.

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3. *(a) Calculate* $\oint_S \boldsymbol{r}\cdot\boldsymbol{n}dS$*, if* S *is the sphere of radius* R *centered at* $x = x_0 > 0$*,* $y = z = 0$*,* $\boldsymbol{n}$ *is the unit outward normal and* $\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$ *is the position vector.*

If $V(\boldsymbol{r})$ *is a scalar field such that* $\nabla^2 V = 0$ *for all* $\boldsymbol{r}$*, where* $\boldsymbol{r}$ *is the position vector in 3D space and* $\boldsymbol{F} = \boldsymbol{\nabla} V$*, calculate, justifying briefly but carefully,*

(b) $\oint_S \boldsymbol{F}\cdot\boldsymbol{n}dS$*, where* S *is any (sufficiently nice) closed surface enclosing the origin.*

(c) $\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$*, where* $\mathcal{C}$ *is any (sufficiently nice) closed curve not passing through the origin.*

This was a gift for many of you! (at last!) See formula (1.72), posted exercises 1.8.5, 1.9.3 and *exam 1 problem 2!*

(a) $\oint_S \boldsymbol{r}\cdot\boldsymbol{n}dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{r}dV = 3V = 4\pi R^3.$

(b) $\oint_S \boldsymbol{F}\cdot\boldsymbol{n}dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{F}dV = \int_V \nabla^2 V dV = 0.$

(c) $\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_A (\boldsymbol{\nabla}\times\boldsymbol{F})\cdot\boldsymbol{n}dA = 0.$

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4. *An n-by-n matrix* $A \equiv [a_{ij}]$ *is said to be orthogonal if* $AA^T = A^TA = I$ *where* I *is the identity matrix.*

(a) Write the orthogonality relationships $AA^T = A^TA = I$ *explicitly in terms of the matrix elements* a_{ij} *using the summation convention and the Kronecker symbol.*

(b) What are the possible values for the $\det(A)$*? Justify.*

(c) Show that the product of two orthogonal matrices is an orthogonal matrix. Justify.

(d) Specify the most general, explicit form of a 2-by-2 orthogonal matrix.

(e) What is the connection between orthogonal matrices and rotation of axes?

See sections 2.6 and 3.3 and posted exercises 3.3.1, 3.3.2, 3.3.12.

(a) $a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij}$.

(b) $\det(A^TA) = \det(I) = 1 = \det(A^T)\det(A) = [\det(A)]^2$, so $\det(A) = \pm 1$.

(c) $(A_1A_2)^T(A_1A_2) = A_2^T(A_1^TA_1)A_2 = I$.

(d) $[\cos\theta, \sin\theta; -\sin\theta, \cos\theta]$

(e) Rotation of coordinate axes correspond to multiplication of the coordinates by an orthogonal matrix.

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5. *How is the triangle linking the complex numbers* $0, 1, 1+i$ *deformed by the mappings (a)* $w = z^2$ *and (b)* $w = e^z$*? Carefully SKETCH both mapped triangles indicating where each vertex is mapped in the* w *plane and specify clearly the shape of each deformed edges. 'Connecting the dots' is not enough. (i.e. state and justify whether an edge becomes a circular arc, a parabolic arc, hyperbolic arc, spiral arc,...). Specify values of deformed angles.*

See posted extra exercises 1 and 2 for section 6.2, also 6.3 extra #2.

OK, you really need to sketch it. Let $A \equiv 0$, $B \equiv 1$, $C \equiv 1+i$ then AB corresponds to $z = x$ and $w_1 = z^2 = x^2$, $w_2 = e^z = e^x$, so
$AB : z = x$ stays on the real axis from 0 to 1 for z^2 but from 1 to e for e^z.
$BC : z = 1 + iy$ so $w_1 = z^2 = 1 - y^2 + 2iy$ is a parabolic arc in w_1 plane, $w_2 = e^z = ee^{iy}$ is a circular arc centered at 0 of radius e and argument from 0 to 1 (in radians).
$AC : z = x + ix$ so $w_1 = 2ix^2$ is pure imaginary from 0 to 2, $w_2 = e^z = e^x e^{ix}$ is a complex number with modulus $r = e^x$ and argument $\theta = x$, so this is a spiral arc $r = e^\theta$.
Pictures left to you. All angles are preserved except for angle at the origin in $w = z^2$ mapping. That angle is doubled from $\pi/4$ to $\pi/2$.

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6. *If $|p| < 1$, what is $\sum_{n=0}^{\infty} p^n \cos nx$? Justify briefly but carefully.*

This was supposed to be a gift! This is posted exercise 6.1.7, we solved it in class at your request! using $\sum_{n=0}^{\infty} p^n \cos nx = \Re\left(\sum_{n=0}^{\infty} p^n e^{inx}\right)$ and the geometric series:

$$\sum_{n=0}^{\infty} p^n e^{inx} = \sum_{n=0}^{\infty} \left(pe^{ix}\right)^n = \frac{1}{1 - pe^{ix}} = \frac{1 - pe^{-ix}}{(1 - p\cos x)^2 + p^2 \sin^2 x} = \frac{1 - p\cos x - ip\sin x}{1 + p^2 - 2p\cos x},$$

so

$$\sum_{n=0}^{\infty} p^n \cos nx = \frac{1 - p\cos x}{1 + p^2 - 2p\cos x}.$$

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7. *(a) Find all the roots of $z^4 + a^4 = 0$ where a is real. (b) Compute the integral $\int_{-\infty}^{\infty} \frac{dx}{a^4 + x^4}$, justifying your steps carefully.*

See Fig 7.4, example 7.2.2 done in class and exercises 7.2.22.

(a) $z^4 \equiv r^2 e^{i4\theta} = -a^4 = a^4 e^{(2k+1)i\pi}$ where k is any integer so $r = a > 0$ and $\theta = (2k+1)\pi/4$. There are 4 distinct roots $z_1 = ae^{i\pi/4}$, $z_2 = iz_1$, $z_3 = -z_1$, $z_4 = -z_2$. Plot them!

(b) Use a contour as in Fig 7.4. There are two simple poles inside. The integral over the large half-circle goes to zero. The two "residues" from the simple poles at z_1 and z_2 are easily calculated if we don't rush into it. First note that $z^4 + a^4 = (z^2 - z_1^2)(z^2 + z_1^2)$ and also $z^4 + a^4 = (z^2 - z_2^2)(z^2 + z_2^2)$, then for the simple pole at z_1

$$\frac{1}{z^4 + a^4} = \frac{f(z)}{z - z_1}, \quad \text{with} \quad f(z) = \frac{1}{(z + z_1)(z^2 + z_1^2)},$$

and by Cauchy's formula this poles contributes

$$2\pi i f(z_1) = \frac{2\pi i}{(z_1 + z_1)(z_1^2 + z_1^2)} = \frac{\pi i}{2z_1^3}.$$

likewise the simple pole at z_2 contributes $\pi i/2z_2^3$, now $z_2 = iz_1$ so the sum of these two contributions is

$$\frac{\pi i}{2z_1^3} + \frac{\pi i}{2z_2^3} = \frac{\pi(i-1)}{2z_1^3} = \frac{\pi\sqrt{2}e^{i3\pi/4}}{2a^3 e^{i3\pi/4}} = \frac{\pi}{\sqrt{2}a^3}.$$

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8. *Calculate*

$$\int_0^{2\pi} \frac{(p\cos\theta - 1)\cos n\theta}{2p\cos\theta - 1 - p^2} d\theta$$

for all integer $n \geq 1$ and $|p| < 1$. Justify briefly but carefully. Does the integral make sense and can you calculate it for any real p?

This is not a trivial problem but it is a minor modification of a problem that was solved explicitly in class. See example 7.2.1 and a very similar exercise done in class namely

$$\int_0^{2\pi} \frac{3\cos(n\theta)}{5 - 4\cos\theta} d\theta$$

(*i.e.* basically the special case of $p = 2$). See also posted exercises 7.2.6, 7.2.7, 7.2.8, 7.2.9.

Let $z = e^{i\theta}$ so $d\theta = -idz/z$ and $2\cos\theta = z + 1/z$, $\cos n\theta = \Re(z^n)$. The integral becomes a complex integral around the unit circle. Clean up the integrand, denominator has two zeros $z_+ = 1/p$, $z_- = p$. Only p is inside unit circle, it's a simple pole. If you rewrite the integrand as $f(z)/(z-p)$, the integral is equal to $2\pi i f(p)$ by Cauchy's formula. Final answer is πp^n. This problem is the reverse of problem 6, *i.e.* knowing the function, find its Fourier Series, and that observation provides another way to deal with this integral by integrating the series term by term.
Yes integral converges for other values of p and can be computed similarly but now $1/p$ is inside the contour and p is outside. The integral vanishes if $p = 1$.