1. *Calculate the area between the curves*

$$pV = T_1, \quad pV = T_2, \qquad T_2 > T_1 > 0,$$
$$pV^\gamma = S_1, \quad pV^\gamma = S_2, \qquad S_2 > S_1 > 0,$$

in the (p, V) plane with $\gamma > 1$. SKETCH the curves, identifying each of them clearly.

See posted exercise and curvilinear coords notes. This is the mechanical energy produced by a perfect gas during a Carnot cycle. Sketch left to you, it's 221 level but very few got it right! Let $T = pV$ and $S = pV^\gamma$, then Area $=\int\int dpdV = \int\int JdTdS$ where J is the Jacobian, i.e.

$$J = \begin{vmatrix} \partial p/\partial T & \partial p/\partial S \\ \partial V/\partial T & \partial V/\partial S \end{vmatrix} = \frac{1}{\begin{vmatrix} \partial T/\partial p & \partial T/\partial V \\ \partial S/\partial p & \partial S/\partial V \end{vmatrix}} = \frac{1}{(\gamma-1)pV^\gamma} = \frac{1}{(\gamma-1)S},$$

and

$$Area = \frac{1}{(\gamma-1)}\int_{T_1}^{T_2} dT \int_{S_1}^{S_2} \frac{dS}{S} = \frac{(T_2 - T_1)}{\gamma - 1}\ln\left(\frac{S_2}{S_1}\right).$$

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2. $\varphi = \varphi(x_1, x_2, x_3)$, $v_k = v_k(x_1, x_2, x_3)$ *are scalar and vector fields that are at least twice continuously differentiable with respect to each coordinate and i, j, k are indices running from 1 to 3. Repeated indices imply summation over the range of that index. ϵ_{ijk} is the permutation tensor (Levi-Civita symbol), δ_{ij} is the identity tensor (Kronecker symbol). Calculate (you must show your steps/reasoning)*

(a) $\epsilon_{132} = -1$, (b) $\epsilon_{ijk}\epsilon_{kji} = -\epsilon_{ijk}\epsilon_{ijk} = -6$,

(c) $\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi = \epsilon_{ikj}\frac{\partial^2\varphi}{\partial x_k \partial x_j} = -\epsilon_{ijk}\frac{\partial^2\varphi}{\partial x_k \partial x_j} = \varphi$, hence it is zero.

(d) $\epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_k = \epsilon_{jik}\frac{\partial^2 v_k}{\partial x_j \partial x_i} = -\epsilon_{ijk}\frac{\partial^2 v_k}{\partial x_i \partial x_j}$, hence zero again. Same as (c) really.

(e) $\epsilon_{ijk}\,\delta_{i1}\,\delta_{2j}\,\delta_{k3} = 1$.

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3. *If $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$ are two right-handed orthonormal bases in $\mathbb{R}^3$ and $\boldsymbol{x} = x_i\boldsymbol{e}_i = x'_i\boldsymbol{e}'_i$ then $x'_i = Q_{ij}x_j$ (summation convention). What is (show/explain your steps)*

(a) $x'i = \boldsymbol{e}'_i \cdot \boldsymbol{x} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j\, x_j$, thus $Q_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$ (b) $Q_{ik}Q_{jk} = \delta_{ij}$ (c) $Q_{ki}Q_{kj} = \delta_{ij}$. (d) $\epsilon_{ijk}Q_{i1}Q_{j2}Q_{k3} = \det(\boldsymbol{Q}) = +1 =$ volume spanned by three orthornormal vectors (and bases have same right-handed orientation).

(e) Find the matrix $[Q_{ij}]$ if $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$ corresponds to a right hand rotation of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ by an angle α about $\boldsymbol{e}_1$.
Make a sketch (see fig 2.20 but sketch basis vectors) then recall that $Q_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$ and

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

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4. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-by-1, $\boldsymbol{A}$ is n-by-n and $()^T$ denotes transpose, state and explain what is

(a) $\det(\boldsymbol{a}^T\boldsymbol{b}) = \boldsymbol{a}^T\boldsymbol{b}$, (b) $\det(\boldsymbol{a}\boldsymbol{b}^T) = 0$, (c) $\det(3\boldsymbol{A}) = 3^n \det(\boldsymbol{A})$, (d) $\det(\boldsymbol{A} - \boldsymbol{A}^T) = \det(\boldsymbol{A} - \boldsymbol{A}^T)^T = \det(\boldsymbol{A}^T - \boldsymbol{A}) = (-1)^n \det(\boldsymbol{A} - \boldsymbol{A}^T) = 0$ if n odd.

(e) $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & a & b \\ 0 & a & b & c \\ a & b & c & d \end{vmatrix} = a^4$, switch rows (or colums) to make it into an upper triangular.

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5. *Find the mirror image of the vector* $\boldsymbol{a} = (1, 2, 3)$ *about the plane perpendicular to the vector* $(4, 1, 2)$.

See your class notes on reflection about a plane, Householder tensor and exercises 6,7 and 8 in tensor handout.

Reflection about a plane perpendicular to the *unit* vector $\boldsymbol{n}$ is given by the tensor

$$\boldsymbol{H} = \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}$$

such that the reflection of vector $\boldsymbol{a}$ is

$$\boldsymbol{H} \cdot \boldsymbol{a} = \boldsymbol{a} - 2\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{a}).$$

Here $\boldsymbol{n} = (4, 1, 2)/\sqrt{4^2 + 1^2 + 2^2} = (4, 1, 2)/\sqrt{21}$, and $\boldsymbol{n} \cdot \boldsymbol{a} = (1, 2, 3) \cdot (4, 1, 2)/\sqrt{21} = (4 + 2 + 6)/\sqrt{21} = 12/\sqrt{21}$ and

$$\boldsymbol{H} \cdot \boldsymbol{a} = \boldsymbol{a} - 2\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{a}) = (1, 2, 3) - \frac{24}{21}(4, 1, 2) = (-75, 18, 15)/21.$$