1. *Consider three vectors* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *in three-dimensional space such that* $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$. *Define* a, b, c *to be the magnitudes of the vectors* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, *respectively, and let* α *be the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$. *Likewise,* β *is the angle between* $\boldsymbol{b}$ *and* $\boldsymbol{c}$, *and* γ *is the angle between* $\boldsymbol{c}$ *and* $\boldsymbol{a}$. *Let* $\boldsymbol{a}\times\boldsymbol{b} = A\hat{\boldsymbol{z}}$ *where* $A>0$ *and* $\hat{\boldsymbol{z}}$ *is a unit vector.*
(a) Sketch the problem.
(b) What is the geometrical interpretation of A *and what is its value in terms of* a, b *and* α*?*
(c) Calculate $\boldsymbol{c}\cdot\hat{\boldsymbol{z}}$. *(d) Calculate* $\boldsymbol{b}\times\boldsymbol{c}$ *and* $\boldsymbol{c}\times\boldsymbol{a}$. *(e) Deduce the 'law of sines'.*

See Exercise 1.3.9 and figure 1.19 on p 28 but watch out for different conventions (to force you to re-derive).
Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form a closed triangle loop. They are all in same plane. If $\boldsymbol{a}\times\boldsymbol{b} = A\hat{\boldsymbol{z}}$ that means they are all orthogonal to $\hat{\boldsymbol{z}}$, hence $\boldsymbol{c}\cdot\hat{\boldsymbol{z}}=0$. A is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$, that is 1/2 the area of the $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ triangle and $A = |\boldsymbol{a}\times\boldsymbol{b}| = ab\sin\alpha$. Now geometrically thinking $\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{c}\times\boldsymbol{a} = \boldsymbol{a}\times\boldsymbol{b} = A\hat{\boldsymbol{z}}$. Algebraically, using the properties of cross-product: $\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{b}\times(-\boldsymbol{a}-\boldsymbol{b}) = -\boldsymbol{b}\times\boldsymbol{a} = \boldsymbol{a}\times\boldsymbol{b}$. Therefore, expressing the magnitudes of these different cross products in terms of the vector magnitudes and the sine of the angle between them, we get

$$ab\sin\alpha = bc\sin\beta = ca\sin\gamma = A$$

or , dividing by abc:

$$\frac{\sin\alpha}{c} = \frac{\sin\beta}{a} = \frac{\sin\gamma}{b} = \frac{A}{abc}.$$

This is the "law of sines" which is usually written differently (as in the book) because of different naming convention for the angles.

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2. *Let* $\boldsymbol{r}$ *represent the position vector in 3D space and* r *its magnitude. The gradient vector operator is denoted* $\boldsymbol{\nabla}$. *If* $f(r)$ *is an arbitrary twice differentiable function of* r, *calculate the following expressions showing or stating the key steps of your reasoning*

(a) $\boldsymbol{\nabla} f(r)$: Example 1.5.5 p 41 + class notes where we deduce the answer quicker from the geometric meaning of the gradient and directional derivatives.

(b) $\boldsymbol{\nabla}\cdot\boldsymbol{r}$: Example 1.6.1 p 44 + class notes

(c) $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f(r) = 0$ ALWAYS even if f was a more general function of the coordinates. See formula (1.92) p 55 and discussion thereafter (*"all gradients are irrotational. Note that zero in (1.92) is a mathematical identity..."*). See also section 1.12 and formula (1.127). Important stuff!

(d) $\boldsymbol{\nabla}\times(f(r)\boldsymbol{r})$: Example 1.7.2 p 48, also by direct application of (a) above: $f(r)\boldsymbol{r} \equiv dF/dr\,\hat{\boldsymbol{r}} = \boldsymbol{\nabla}F(r)$, so $\boldsymbol{\nabla}\times(f(r)\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}F(r) = 0$ by (c).

(e) $\int_V \boldsymbol{\nabla}^2 f(r) dV$, where V is the sphere of radius R centered at the origin: This integral could be done directly by computing the volume integral using Example 1.8.1 p 54 but it is easier to use the **Divergence (Gauss) theorem**:

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{v}\, dV = \oint_S \boldsymbol{v}\cdot\hat{\boldsymbol{n}} dS$$

for any *closed* volume V where S is the surface of V and $\hat{\boldsymbol{n}}$ is the unit normal pointing outward. Then in our case $\int_V \boldsymbol{\nabla}^2 f(r) dV = \int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f(r)) dV = \oint_S (\boldsymbol{\nabla} f)\cdot\hat{\boldsymbol{n}}\, dS$ and, by (a), $\boldsymbol{\nabla} f(r) = df/dr\,\hat{\boldsymbol{r}}$. Now, for any sphere centered at the origin, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, so the integral is simply

$$\int_V \boldsymbol{\nabla}^2 f(r) dV = \left.\frac{df}{dr}\right|_{r=R} \oint_S dS = 4\pi R^2 f'(R).$$

See formula (1.90) for definition of Laplacian operator: $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$. See also exercise 1.10.2 p 70 and sections 1.12 and 1.13 on Potential theory and Gauss's law whose summary is $\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \Rightarrow \boldsymbol{E} = -\boldsymbol{\nabla}\varphi$, so $\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\boldsymbol{\nabla}^2\varphi$ and

$$\int_V \boldsymbol{\nabla}^2\varphi \, dV = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{E} dV = -\oint \boldsymbol{E} \cdot d\boldsymbol{S} = -\int_V \frac{\rho}{\epsilon_0} dV. \tag{1.142}$$

The last equality is known as Gauss's law in electromagnetism. As this is true for any closed volume V, this implies $\boldsymbol{\nabla}^2\varphi = -\rho/\epsilon_0$ (eqn. 1.145, p 84). This is a *Poisson equation*. Its solution gives the electric potential φ if the charge density ρ is known. Poisson equations pop up in all sorts of places, in flow of an incompressible fluid for instance.

(f) $\oint_S \boldsymbol{r} \cdot d\boldsymbol{S}$ where S is the surface of the sphere of radius 1 centered at $x_c = 5$, $y_c = z_c = 0$.

By Gauss: $\oint_S \boldsymbol{r} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{r} dV = 4\pi$ using (b) above.

(g) $\oint_C \boldsymbol{r} \times d\boldsymbol{r}$ where C is the curve $(x - x_0)^2/a^2 + y^2/b^2 = 1$, $z = 0$.

$\oint_C \boldsymbol{r} \times d\boldsymbol{r} = 2\pi ab$ which is twice the area of the ellipse. See problem 1.11.1 p 75. Also we did this in class when discussing motion of planets. See extra exercises posted on web page.

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3. *Consider the vector field $\boldsymbol{v} = v(r)\hat{\boldsymbol{\varphi}}$ where r is the distance to the origin and $\hat{\boldsymbol{\varphi}}$ is the unit vector in the azimuthal direction in spherical coordinates.*

(a) Calculate $\int_{NHS} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S}$ where NHS is the surface of the Northern hemisphere of a sphere of radius R i.e. the surface $x^2 + y^2 + z^2 = R^2$, $z \geq 0$ and $d\boldsymbol{S}$ is the surface element pointing out of the sphere.

(b) Calculate $\int_{EHS} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S}$ where EHS is the surface of the Eastern Hemisphere of the sphere of radius R, i.e. the surface $x^2 + y^2 + z^2 = R^2$, $y \geq 0$.

(c) Calculate $\oint_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S}$ where S is the surface of the "smoothed cube" $x^8 + y^8 + z^8 = R^8$.

Three direct applications of **Stokes Theorem** which says that for any orientable surface S:

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r}$$

where C is the curve boundary of the surface S and the orientation of the line integral and the surface element $d\boldsymbol{S}$ must obey the right-hand rule (because of the definition of the curl).

Then (a) $\int_{NHS} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = \oint_{equator} v(r)\hat{\boldsymbol{\varphi}} \cdot d\boldsymbol{r}$ but the line element along the equator is $d\boldsymbol{r} = \hat{\boldsymbol{\varphi}} R d\varphi$ so the integral is simply $= 2\pi R v(R)$.

(b) $\int_{EHS} \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = \oint_{meridian} v(r)\hat{\boldsymbol{\varphi}} \cdot d\boldsymbol{r}$ where the meridian is the reference meridian *i.e.* the circle $x^2 + z^2 = R^2$ with $y = 0$. On any meridian $\hat{\boldsymbol{\varphi}} \cdot d\boldsymbol{r} = 0$, so the integral is zero.

(c) The equation $x^8 + y^8 + z^8 = R^8$ obviously "looks like" the equation of a sphere but because of the higher powers this CLOSED surface looks more like a smoothed cube (try plotting the "smoothed square" $x^8 + y^8 = R^8$). In any case the key fact is that it is closed and therefore $\oint_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S} = 0$. See Exercise 1.10.1 p 70 which uses Gauss's theorem and $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} = 0$ (another important identity), also Exercise 1.11.4.

Gauss and **Stokes Theorems** are the **Fundamental theorems of Vector Calculus** relating divergence, curl, volume integral, surface integral and line integrals. There are several versions of these theorems that you can derive using special vector fields (*e.g.* try $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ in the divergence theorem). These theorems also give "physical" interpretations of divergence as the flux (or "flow") through a closed surface normalized by volume and curl as the "circulation" of a vector field around a closed loop, normalized by area.

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4. *Pick appropriate coordinates then specify explicit integral formulas for the area and volume of a torus (i.e. specify the variables and the bounds of integration). You may, but not do have to, compute the integrals.*

We sketched this in class the Friday before the exam. See notes for a figure. The parametrization of the torus is

$$x = (R_1 + r\cos\theta)\cos\varphi, \quad y = (R_1 + r\cos\theta)\sin\varphi, \quad z = r\sin\theta,$$

where $0 \le r \le R_2$ and θ and φ both run from 0 to 2π IF $R_1 > R_2$ which is required for a torus (otherwise it's an apple not a donut!).

The position vector is $\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$. We need the displacement vectors

$$\frac{\partial \boldsymbol{r}}{\partial r} = \hat{\boldsymbol{x}}\cos\theta\cos\varphi + \hat{\boldsymbol{y}}\cos\theta\sin\varphi + \hat{\boldsymbol{z}}\sin\theta, \Rightarrow h_r = 1$$

$$\frac{\partial \boldsymbol{r}}{\partial \varphi} = -\hat{\boldsymbol{x}}(R_1 + r\cos\theta)\sin\varphi + \hat{\boldsymbol{y}}(R_1 + r\cos\theta)\cos\varphi, \Rightarrow h_\varphi = R_1 + r\cos\theta$$

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = -\hat{\boldsymbol{x}}\, r\sin\theta\cos\varphi - \hat{\boldsymbol{y}}\, r\sin\theta\sin\varphi + \hat{\boldsymbol{z}} r\cos\theta, \Rightarrow h_\theta = r.$$

It is straightforward to verify that these vectors are orthogonal. Hence the surface element is

$$dA = \left|\frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial \theta}\right| d\varphi d\theta = h_\varphi h_\theta d\varphi d\theta = R_2(R_1 + R_2\cos\theta) d\varphi d\theta$$

and the volume element is

$$dV = \left|\frac{\partial \boldsymbol{r}}{\partial r} \cdot \frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial \theta}\right| dr d\varphi d\theta = h_r h_\varphi h_\theta dr d\varphi d\theta = r(R_1 + r\cos\theta) dr d\varphi d\theta.$$

So the surface area is

$$A = \int_0^{2\pi}\int_0^{2\pi} R_2(R_1 + R_2\cos\theta) d\varphi d\theta = 4\pi^2 R_1 R_2$$

and the volume

$$V = \int_0^{R_2}\int_0^{2\pi}\int_0^{2\pi} r(R_1 + r\cos\theta) d\varphi d\theta dr = 2\pi^2 R_1 R_2^2.$$

Note that both of these formula simply corresponds to the cross-sectional perimeter $2\pi R_2$ and area πR_2^2 multiplied by the perimeter of the centerline of the torus $2\pi R_1$. This says something about the geometry of the torus, can you see what?