Exercises

1. The definition of the alternating, or permutation (or *Levi-Civita*), symbol is

$$\epsilon_{j_1 j_2 \dots j_n} \equiv \pm 1, \quad \text{or } 0,$$

depending on whether $(j_1, j_2, \dots, j_n)$ is an *even* ($\epsilon = +1$) or *odd* ($\epsilon = -1$) permutation of $(1, 2, 3, \dots, n)$. For instance $\epsilon_{1,2,\dots,n} = +1 = \epsilon_{2,3,1,4,\dots,n}$ but $\epsilon_{2,1,3,4\dots,n} = -1 = \epsilon_{1,3,2,4,\dots,n}$. The permutation symbol is 0 if $(j_1, j_2, \dots, j_n)$ is not a permutation of $(1, 2, 3, \dots, n)$. In particular, it is zero if any of the indices are repeated *e.g.* $\epsilon_{1,1,3,4,\dots,n} = 0$. We assume implicitly that the range of each index j_i, $i = 1, \dots, n$ is $1, \dots, n$.
(a) Expand out explicitly $\epsilon_{ij}\, a_i b_j$, $i, j = (1, 2)$ (sum over repeated indices!).
(b) Expand out explicitly $\epsilon_{ijk}\, a_i b_j c_k$, $i, j, k = (1, 2, 3)$.
(c) Expand out explicitly $\epsilon_{ijkl}\, a_i b_j c_k d_l$, $i, j, k, l = (1, 2, 3, 4)$.

2. We demonstrated in class (W 10/29/2003) how to calculate a 4-by-4 determinant using multiple shearings of the 4-dimensional parallelepiped to "rectify" it, *i.e.* transform it into a parallelepiped whose edges are aligned with the natural basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$. We then realized that we had everything needed to compute the determinant when it was in "triangular" form. Use the same strategy to calculate *explicitly* the determinant

$$D \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

i.e. calculate b_2', c_3'' and D. Compare with exercise 1.(b). Determine the minimum number of arithmetic operations needed to calculate D this way and the 1.(b) way. Can you generalize this arithmetic operation count for a n-by-n determinant?

3. In exercise 2, what happens to $D \equiv \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ if we do several shearings "at once" *i.e.* replace $\boldsymbol{b}$ by $\boldsymbol{b} + \beta\boldsymbol{a}$ and $\boldsymbol{c}$ by $\boldsymbol{c} + \gamma\boldsymbol{a}$ "simultaneously"? Imagine we have several 'computers' to do the job and we let computer 1 modify $\boldsymbol{b}$ and computer 2 modify $\boldsymbol{c}$, simultaneously (or "in parallel"). What would happen to D if computer 1 made the transformation $\boldsymbol{b} \to \boldsymbol{b} + \beta\boldsymbol{a}$ while computer 2 made the transformation $\boldsymbol{c} \to \boldsymbol{c} + \gamma\boldsymbol{b}$? What if computer 1 transformed $\boldsymbol{b} \to \boldsymbol{b} + \beta\boldsymbol{a}$ but computer 2 did $\boldsymbol{a} \to \boldsymbol{a} + \alpha\boldsymbol{b}$?

4. Calculate

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

5. Calculate

$$\begin{vmatrix} 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & a_2 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-1} \\ a_n & 0 & \cdots & \cdots & 0 \end{vmatrix}.$$

[Hint: if you don't "see it", start with $n = 2$, then $n = 3$, then $n = 4$, first.]

6. What is the area of the triangle whose 3 vertices have the coordiantes (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)? What is the volume of the tetrahedron whose 4 vertices have the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)?

7. Write a real or pseudo-code to compute the determinant of an n-by-n matrix $[a_{ij}]$, $i, j = 1, \ldots, n$ using shearings *and* vector swaps (orientation rule), as needed in case one of the “pivots” is zero. Try it out on your calculator or computer.

8. Show that for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$,

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{a} & \boldsymbol{a} \cdot \boldsymbol{b} \\ \boldsymbol{b} \cdot \boldsymbol{a} & \boldsymbol{b} \cdot \boldsymbol{b} \end{vmatrix}.$$

The right hand side can be generalized to any dimension allowing us to calculate the area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ in any dimension.