Our concept of vectors can be generalized to n-dimensions. The vector space $\mathbb{R}^n$ consists of the ordered set of n-tuplets of real numbers $\boldsymbol{x} \equiv (x_1, x_2, \dots, x_n)$. These elements can be added by adding respective components:

$$\boldsymbol{x} + \boldsymbol{y} \equiv (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = \boldsymbol{y} + \boldsymbol{x} \tag{1}$$

This corresponds to the parallelogram addition rule in 2D and 3D.
Vectors can be multiplied by a scalar $\alpha \in \mathbb{R}$:

$$\alpha \boldsymbol{x} \equiv (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \tag{2}$$

Those two operations, addition of elements and multiplication by a scalar, are the key ingredients that define a *vector space*.
The dot product is a scalar given by

$$\boldsymbol{x} \cdot \boldsymbol{y} \equiv x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \tag{3}$$

The norm (*i.e.* size) of vector $\boldsymbol{x}$ is defined by the positive number

$$||\boldsymbol{x}|| \equiv \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}. \tag{4}$$

The dot product allows us to define the angle θ between two n-dimensional vectors:

$$\cos\theta \equiv \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||\, ||\boldsymbol{y}||}. \tag{5}$$

Then two vectors are *orthogonal* if their dot product is zero, $\boldsymbol{x} \cdot \boldsymbol{y} = 0 \Leftrightarrow \boldsymbol{x} \perp \boldsymbol{y}$.

How do we measure volumes in an n-dimensional space? with determinants! Let's review what we know about them in 2D and 3D, then generalize to nD.

In **2D-space** $\mathbb{R}^2$:

$$\det(\boldsymbol{a}, \boldsymbol{b}) \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}. \tag{6}$$

This *scalar* that we call determinant represent the *signed area* of the parallelogram spanned by $\boldsymbol{a}, \boldsymbol{b}$. $\mathrm{Det}(\boldsymbol{a}, \boldsymbol{b}) = Area$, if the orientation of $\boldsymbol{a}, \boldsymbol{b}$ corresponds to the natural orientation of the space *i.e.* $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ and counter-clockwise rotation to go from $\boldsymbol{a}$ to $\boldsymbol{b}$. $\mathrm{Det}(\boldsymbol{a}, \boldsymbol{b}) = -Area$, if $\boldsymbol{a}, \boldsymbol{b}$ has opposite orientation, *i.e.* clockwise rotation to go from $\boldsymbol{a}$ to $\boldsymbol{b}$. *Make your own picture!!*

In **3D-space** $\mathbb{R}^3$: The signed volume is given by the mixed (or "triple scalar") product: $\pm V = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. This is the volume V of the parallelepiped spanned by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ if those three vectors have the same orientation (*i.e.* right-handed) as $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, it is $-V$ otherwise (left-handed). The mixed product has the well-known (?) geometric properties of invariance under cyclic rotation of the vectors and commutativity of the dot product, $\boldsymbol{d} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{d}$, and anti-commutativity, $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$: of the cross-product

$$\begin{aligned} &(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b} = \\ &\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \\ &-\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b}) = -\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) = -\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = \\ &-(\boldsymbol{b} \times \boldsymbol{a}) \cdot \boldsymbol{c} = -(\boldsymbol{c} \times \boldsymbol{b}) \cdot \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b} \end{aligned} \tag{7}$$

The mixed product can be expressed as a 3-by-3 determinant in terms of the Cartesian components of the vectors and that determinant can be evaluated in various ways

$$\det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$

$$= \epsilon_{ijk} a_i b_j c_k = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_2 b_1 c_3 - a_1 b_3 c_2 - a_3 b_2 c_1 \tag{8}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \textit{ etc.}$$

In n**D-space** $\mathbb{R}^n$, we can define the signed volume spanned by the n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ as the determinant $D \equiv \det(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) = \pm V$ through the geometric properties of
(i) Parallel shearing does not change volume: Parallel shearing the n-parallelepiped corresponds to adding to one vector, $\boldsymbol{a}_k$ say, a linear combination of the other vectors $\boldsymbol{a}_k \to \widetilde{\boldsymbol{a}}_k = \boldsymbol{a}_k + \sum_{i\neq k} \alpha_i \boldsymbol{a}_i$ *keeping all other vectors the same.* The determinant (signed volume) does not change under such a parallel shearing ("base" and "height" remain the same):

$$\det(\boldsymbol{a}_1, \ldots, \widetilde{\boldsymbol{a}}_k, \ldots, \boldsymbol{a}_n) = \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_n), \quad \forall \alpha_i, k, \boldsymbol{a}_j.$$

(ii) *Stretching one vector by factor α, stretches volume by factor α*:

$$\det(\boldsymbol{a}_1, \ldots, \alpha\boldsymbol{a}_k, \ldots, \boldsymbol{a}_n) = \alpha \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_n), \quad \forall \alpha, k, \boldsymbol{a}_j.$$

These two properties (i) and (ii) can be combined into a single *shearing and stretching* rule. If one and only one vector, $\boldsymbol{a}_k$ say, is replaced by a linear combination of all the vectors, then the determinant is multiplied by the coefficient of $\boldsymbol{a}_k$:

1. Shearing and Stretching Rule

If $\widetilde{\boldsymbol{a}}_k = \sum_{i=1}^n \alpha_i \boldsymbol{a}_i$ then

$$\det(\boldsymbol{a}_1, \ldots, \widetilde{\boldsymbol{a}}_k, \ldots, \boldsymbol{a}_n) \equiv \alpha_k \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_n), \quad \forall \alpha_i, k, \boldsymbol{a}_j. \tag{9}$$

2. Orientation Rule

The determinant changes sign if any two vectors are permuted:

$$\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_l, \ldots, \boldsymbol{a}_n) \equiv -\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_n), \quad \forall k, l, \boldsymbol{a}_j. \tag{10}$$

3. Volume of Natural Basis

The 'natural basis' for $\mathbb{R}^n$ consists of the n vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, where $\boldsymbol{e}_i$ has all components zero except for the i-th component which is 1, *i.e.* $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)$, etc. These are orthogonal vectors of magnitude 1. The signed volume spanned by these vectors is

$$\det(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{vmatrix} \equiv 1. \tag{11}$$

These three rules imply many other properties but they also fully specify the determinant. The basic idea is to use rules (9) and (10) to transform the determinant into a multiple of $\det(\boldsymbol{e}_1, ..., \boldsymbol{e}_n)$. Then by (11), the multiplication factor is in fact the determinant.
Two useful properties that can be deduced from these three rules are:

(P1) If two vectors are equal then the determinant is zero (by rule (10), permuting the two identical vectors leaves the det unchanged *and* changes its sign, so it must be zero).

(P2) If one vector is identically zero (*i.e.* all its components are zero), the determinant is zero (by (9), add any vector to the zero vector, then two vectors are identical, hence by (P1) it is zero).

(P3) If the n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ do not form a basis for $\mathbb{R}^n$, *i.e.* if they are "co-planar", then their det (signed volume) is zero (by (9) $\widetilde{\boldsymbol{a}}_k$ can be made to vanish for some k, then by (P2) det=0).

The shearing and stretching rule (9) can be replaced by the more fundamental *multi-linearity rule.* This rule can be induced from geometry also (as well as from algebraic properties of cross and mixed products *e.g.* $(\alpha\boldsymbol{a}+\beta\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = \alpha\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{d})+\beta\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})$, etc.). This is the rule (property) that the determinant is a *multi-linear function of the vectors*, *i.e.*

1'. Multi-linearity

If $\boldsymbol{a}_k = \sum_{i=1}^n \beta_i \boldsymbol{b}_i$, for any $k, \beta_i, \boldsymbol{b}_i$, then

$$\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots, \boldsymbol{a}_n) \equiv \sum_{i=1}^n \beta_i \ \det(\boldsymbol{a}_1, \ldots, \boldsymbol{b}_i, \ldots, \boldsymbol{a}_n), \tag{12}$$

where each of the $\boldsymbol{b}_i$ are replacing vector $\boldsymbol{a}_k$ in the n determinants on the right-hand side and all other vectors are identical in all determinants on both sides. It is clear that multi-linearity and (P1) imply the shearing and stretching rule (9). The converse is also true but it's less obvious. The multi-linearity rule, together with the orientation rule and (P1), includes shearing, stretching *and* "parallel breaking."

Further useful properties of determinants are best expressed using matrices. We can view the n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ as the n columns of a matrix A and the latter can be viewed as a "row of columns" or a "column of rows":

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \cdots \boldsymbol{a}_n] \equiv \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_n \end{bmatrix} \tag{13}$$

where

$$\boldsymbol{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \quad \text{and} \quad \boldsymbol{r}_i = [a_{i1} \ \ a_{i2} \ \cdots \ a_{in}]\,. \tag{14}$$

Note the square brackets for the matrix, *i.e.* the table of n^2 elements, and the vertical bars for the determinant, *i.e.* the scalar representing signed volume. Those bits of horizontal lines on the ends of the vertical bars mean a lot!
Some key determinant properties are then

(P4) The determinant of a triangular matrix is the product of the diagonal elements. A triangular matrix is a matrix for which all the elements above or below the diagonal are zero.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}. \tag{15}$$

This can be proved by application of the three principal rules.

(P5) The determinant of the columns is equal to the determinant of the rows

$$\det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \det(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n). \tag{16}$$

In matrix language, this is $\det(A) = \det(A^T)$, the determinant of a matrix equals the determinant of its transpose. This is an important and deep property. It means that all the rules and properties apply to the rows as well as to the columns (but rows and columns cannot be mixed up randomly!). The transpose of A is the matrix obtained by permuting rows and columns (keeping the order!), with A as in (13):

$$A^T \equiv \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}. \tag{17}$$

Finally, there are two explicit formula for determinants that generalize (8). These are of theoretical interest. They are useless for computations as soon as n is larger than about 3. The most explicit formula is the

Permutation formula:

$$\det(A) = \epsilon_{j_1 j_2 \cdots j_n}\, a_{j_1 1} a_{j_2 2} \cdots a_{j_n n} = \epsilon_{j_1 j_2 \cdots j_n}\, a_{1 j_1} a_{1 j_2} \cdots a_{n j_n}, \tag{18}$$

where sums over repeated indices are implicit and $\epsilon_{j_1 j_2 \cdots j_n} = \pm 1$ depending on whether $(j_1, j_2, \ldots, j_n)$ is an even or odd permutation of $(1, 2, \ldots, n)$. Note that this sum as $n!$ terms!

Another useful formula is recursive. It determines an n-by-n determinant in terms of $(n-1)$-by-$(n-1)$ determinants.

Co-factor formula:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, \tag{19}$$

no matter what i or j is selected (no sums over i or j here!). The co-factor C_{ij} is defined as $(-1)^{i+j}$ times the $(n-1)$-by-$(n-1)$ determinant obtained by suppressing row i and column j from A.

These formulas were invented so the people who love to "plug-and-chug" can stay happy forever. Try computing a 100-by-100 determinant using either of these formulas. Yes, you can use your calculator but you must use these formula. If your calculator knows how to compute determinants, you better hope that it uses rules 1, 2, 3 instead of these explicit formulas.

Exercises

1. The definition of the alternating, or permutation (or *Levi-Civita*), symbol is

$$\epsilon_{j_1 j_2 \ldots j_n} \equiv \pm 1, \quad \text{or } 0,$$

depending on whether $(j_1, j_2, \ldots, j_n)$ is an *even* ($\epsilon = +1$) or *odd* ($\epsilon = -1$) permutation of $(1, 2, 3, \ldots, n)$. For instance $\epsilon_{1,2,\ldots,n} = +1 = \epsilon_{2,3,1,4,\ldots,n}$ but $\epsilon_{2,1,3,4\ldots,n} = -1 = \epsilon_{1,3,2,4,\ldots,n}$. The permutation symbol is 0 if $(j_1, j_2, \ldots, j_n)$ is not a permutation of $(1, 2, 3, \ldots, n)$. In particular, it is zero if any of the indices are repeated *e.g.* $\epsilon_{1,1,3,4,\ldots,n} = 0$. We assume implicitly that the range of each index j_i, $i = 1, \ldots, n$ is $1, \ldots, n$.
(a) Expand out explicitly $\epsilon_{ij}\, a_i b_j$, $i, j = (1, 2)$ (sum over repeated indices!).
(b) Expand out explicitly $\epsilon_{ijk}\, a_i b_j c_k$, $i, j, k = (1, 2, 3)$.
(c) Expand out explicitly $\epsilon_{ijkl}\, a_i b_j c_k d_l$, $i, j, k, l = (1, 2, 3, 4)$.

2. We demonstrated in class (W 10/29/2003) how to calculate a 4-by-4 determinant using multiple shearings of the 4-dimensional parallelepiped to "rectify" it, *i.e.* transform it into a parallelepiped whose edges are aligned with the natural basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$. We then realized that we had everything needed to compute the determinant when it was in "triangular" form. Use the same strategy to calculate *explicitly* the determinant

$$D \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

i.e. calculate b_2', c_3'' and D. Compare with exercise 1.(b). Determine the minimum number of arithmetic operations needed to calculate D this way and the 1.(b) way. Can you generalize this arithmetic operation count for a n-by-n determinant?

3. In exercise 2, what happens to $D \equiv \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ if we do several shearings "at once" *i.e.* replace $\boldsymbol{b}$ by $\boldsymbol{b} + \beta \boldsymbol{a}$ and $\boldsymbol{c}$ by $\boldsymbol{c} + \gamma \boldsymbol{a}$ "simultaneously"? Imagine we have several 'computers' to do the job and we let computer 1 modify $\boldsymbol{b}$ and computer 2 modify $\boldsymbol{c}$, simultaneously (or "in parallel"). What would happen to D if computer 1 made the transformation $\boldsymbol{b} \to \boldsymbol{b} + \beta \boldsymbol{a}$ while computer 2 made the transformation $\boldsymbol{c} \to \boldsymbol{c} + \gamma \boldsymbol{b}$? What if computer 1 transformed $\boldsymbol{b} \to \boldsymbol{b} + \beta \boldsymbol{a}$ but computer 2 did $\boldsymbol{a} \to \boldsymbol{a} + \alpha \boldsymbol{b}$?

4. Calculate

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

5. Calculate

$$\begin{vmatrix} 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & a_2 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-1} \\ a_n & 0 & \cdots & \cdots & 0 \end{vmatrix}.$$

[Hint: if you don't "see it", start with $n = 2$, then $n = 3$, then $n = 4$, first.]

6. What is the area of the triangle whose 3 vertices have the coordiantes (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)? What is the volume of the tetrahedron whose 4 vertices have the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)?

7. Write a real or pseudo-code to compute the determinant of an n-by-n matrix $[a_{ij}]$, $i,j = 1,\ldots,n$ using shearings *and* vector swaps (orientation rule), as needed in case one of the "pivots" is zero. Try it out on your calculator or computer.

8. Show that for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$,

$$|\boldsymbol{a}\times\boldsymbol{b}|^2 = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{a} & \boldsymbol{a}\cdot\boldsymbol{b} \\ \boldsymbol{b}\cdot\boldsymbol{a} & \boldsymbol{b}\cdot\boldsymbol{b} \end{vmatrix}.$$

The right hand side can be generalized to any dimension allowing us to calculate the area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ in any dimension.