1. [20 pts] *Consider the differential equation* $\frac{d^2Q}{dt^2} + \gamma\frac{dQ}{dt} + Q = \sin\omega t$ *where* γ *and* ω *are constants with* $0 < \gamma < 1$.

(a) Find a particular solution: Constant coeffs so we try $Q_p(t) \propto \sin\omega t$ but the dQ/dt brings in a $\cos\omega t$ also, so try $Q_p(t) = A\cos\omega t + B\sin\omega t$, substituting in the $Q(t)$ diff eq we get

$$(-\omega^2 A + \gamma\omega B + A)\cos\omega t + (-\omega^2 B - \gamma\omega A + B)\sin\omega t = \sin\omega t$$

now *sin* ans *cos* are linearly independent functions so the equation implies that we need

$$\begin{cases} (1-\omega^2)A + \gamma\omega B = 0 \\ (1-\omega^2)B - \gamma\omega A = 1 \end{cases}$$

The letters γ and ω stand for *parameters* of the problem, *i.e.* although their numerical value is not specified they're assumed to be known and we want to know the solution in terms of those parameters. A and B are undetermined constants that we have introduced to solve the problem, those are the guys we want to solve for in terms of γ and ω. That's a little bit of algebra: the first eqn says $A = \gamma\omega B/(\omega^2-1)$ then the 2nd eqn gives $[(1-\omega^2)^2+\gamma^2\omega^2]B = (1-\omega^2)$ or

$$A = \frac{-\gamma\omega}{(1-\omega^2)^2+\gamma^2\omega^2} \qquad B = \frac{(1-\omega^2)}{(1-\omega^2)^2+\gamma^2\omega^2}\ .$$

(b) Find the general solution: The equation is *linear* in Q so we can use superposition: $Q(t) = CQ_1(t) + DQ_2(t) + Q_p(t)$ where $Q_p(t)$ is the particular solution found in (a) and Q_1, Q_2 are two linearly independent solutions of the homogeneous eqn $\frac{d^2Q}{dt^2} + \gamma\frac{dQ}{dt} + Q = 0$. This eqn has constant coefficients hence we look for $Q(t) = e^{rt}$ for some constant r. This yields the algebraic eqn $r^2 + \gamma r + 1 = 0$ whose solutions are

$$r_1 = \frac{-\gamma + i\sqrt{4-\gamma^2}}{2} \qquad r_2 = \frac{-\gamma - i\sqrt{4-\gamma^2}}{2}$$

where I have used $0 < \gamma < 1$ and $i^2 = -1$. So the general solution is

$$Q(t) = Ce^{-\gamma t/2}\cos\omega_0 t + De^{-\gamma t/2}\sin\omega_0 t + Q_p(t)$$

where $Q_p(t)$ is as in (a), C and D are constants to be determined from the initial conditions and $\omega_0 = \sqrt{1-\gamma^2/4}$.

(c) What is the form of the solution after a long time? well $\gamma > 0$ so $e^{-\gamma t/2} \to 0$ as $t \to +\infty$, therefore $Q(t) \to Q_p(t)$. The systems "forgets" its initial conditions and the solution will eventually look mostly like the particular solution (forced response).

(d) What does "long time" mean for *this* equation in practice? $t \to \infty$ is a pretty long time. The Sun will have shut off long before that and no one will be around to see what happens to $Q(t)$. However we only need to wait until $\exp(-\gamma t/2)$ is "small enough" and that happens pretty quickly on a time scale of $O(1/\gamma)$. For instance $\exp(-10) \approx 4.54\ 10^{-5}$ and that happens at $t = 20/\gamma$. If you're willing to wait twice as long then $\exp(-20) \approx 2.06\ 10^{-9}$. The key result is that the time is inversely proportional to γ.

2. *[20 pts] Find the general solution to* $\frac{d^3y}{dt^3} - y = t$ *then set-up, but do not solve, the system of algebraic equations that would yield the solution with* $y(0) = 1$ *and* $y'(0) = y''(0) = 0$.
The equation is linear in $y(t)$ so the general solution is the sum of the general homogeneous solution and a particular solution. The later is obviously $y_p(t) = -t$. The equation has constant coeffs so try $y(t) = e^{rt}$ as before. That yields $r^3 = 1$ not to be confused with $(r-1)^3 = 0$. The latter would give a repeated triple root but the former as three distinct roots, one real $r_1 = 1$ and a complex conjugate pair $r_2 = r_3^*$.
To find all those roots let's emphasize that we're really in complex space by writing $r^3 = 1$ as $r^3 = e^{2ik\pi}$ where $k = 0, \pm1, \pm2, \ldots$ is any integer. Writing the complex number r in its polar representation $r = |r| \exp(i\theta)$ so that $r^3 = |r|^3 \exp(3i\theta) = e^{2ik\pi}$, we need $|r| = 1$ and $3\theta = 2k\pi$. This yields

$$r = e^{2ik\pi/3}, \qquad k = 0, \pm1, \pm2, \cdots$$

which corresponds to three distinct roots $r_1 = 1$, $r_2 = \exp(2i\pi/3)$ and $r_3 = \exp(-2i\pi/3)$. All other values of k yield copies of these roots. You can get around this by factoring $r^3 - 1 = (r-1)(r^2 + r + 1)$ but would do well to understand the 'complex' approach. (Exercises: find the roots of $r^{17} = 1$; find the roots of $r^9 = 8$).
So the general solution is

$$y(t) = Ae^{r_1 t} + Be^{r_2 t} + Ce^{r_3 t} - t.$$

Imposing the initial conditions leads to the algebraic system

$$\left\{ \begin{array}{ccccc} A & + & B & + & C = 1 \\ r_1 A & + & r_2 B & + & r_3 C = 1 \\ r_1^2 A & + & r_2^2 B & + & r_3^2 C = 0 \end{array} \right. .$$

This system has a unique solution because the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ r_1 & r_2 - r_1 & r_3 - r_1 \\ r_1^2 & r_2^2 - r_1^2 & r_3^2 - r_1^2 \end{vmatrix} = (r_2 - r_1)(r_3 - r_1) \begin{vmatrix} 1 & 0 & 0 \\ r_1 & 1 & 1 \\ r_1^2 & r_2 - r_1 & r_3 - r_1 \end{vmatrix}$$

$$= (r_2 - r_1)(r_3 - r_1) \begin{vmatrix} 1 & 0 & 0 \\ r_1 & 1 & 0 \\ r_1^2 & r_2 - r_1 & r_3 - r_2 \end{vmatrix} = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2) \neq 0$$

as the roots are distinct.

3. [10 pts] The eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$$

are $\lambda = 2i$ with $\mathbf{v} = [2, i]^T$ and their complex conjugate.