1. *For what values of a and b does the following system have (a) a unique solution, (b) no solution, (c) infinitely many solutions? (specify ALL the a and b values, not some arbitrary a and b that you picked because they're your favorite numbers!). Draw a sketch illustrating each case* **for this system**.

$$\begin{aligned} 3x + 2y &= a \\ 6x + by &= 6 \end{aligned}$$

Gaussian elimination: Row 2 := Row 2 - 2 Row 1, so

$$\begin{aligned} 3x + 2y = &\quad a \\ (b-4)y = &\quad 6 - 2a \end{aligned}$$

Therefore there is a unique solution for any a if $b \neq 4$. There is an infinity of solution if $b = 4$ and $a = 3$. There are no solutions if $b = 4$ and $a \neq 3$.
Graphically, the first equation $3x + 2y = a$ is the line that goes through $(x = a/3, y = 0)$ and $(x = 0, y = a/2)$, the second is the line going through $(x = 1, y = 0)$ and $(x = 0, y = 6/b)$. The lines intersect at one point if $b \neq 4$. They are parallel if $b = 4$ and distinct if $a \neq 3$. Both equations represent the same line if $b = 4$ and $a = 3$.

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2. *The matrix* $\mathbf{A}$ *is m-by-p, the matrix* $\mathbf{B}$ *is m-by-q, where m, p and q are distinct positive integers, so* $m \neq p \neq q \neq m$. *Specify the dimensions of the matrix* $\mathbf{C}$ *or the appropriate error message (i.e. why there is something wrong with the operation).*

(a) $\mathbf{C} = \mathbf{AB}$ Error: inner matrix dimensions must agree.

(b) $\mathbf{C} = \mathbf{BA}$ Error: inner matrix dimensions must agree.

(c) $\mathbf{C} = \mathbf{B^T A}$ is a q-by-p matrix.

(d) $\mathbf{C} = \mathbf{A^T A}$ is a p-by-p matrix.

(e) $\mathbf{C} = \mathbf{A^T A^{-1}}$ Error: A is not square, hence it does not have an inverse.

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3. *The n-by-n square matrix* $\mathbf{A}$ *is called* **orthogonal** *if* $\mathbf{A^T} = \mathbf{A^{-1}}$. *Let* $\mathbf{b}$ *be an arbitrary n-by-1 vector. Write down a few steps to show your reasoning.*

(a) $\mathbf{A(A^T b) = AA^T b = AA^{-1} b = b}$

(b) $\mathbf{(A^T b)^T A = b^T AA = b^T A^2}$

(c) $\det(\mathbf{A}) = \det(\mathbf{A^T}) = \det(\mathbf{A^{-1}}) = \mathbf{1}/\det(\mathbf{A})$ Hence $\det(\mathbf{A}) = \pm\mathbf{1}$.

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4. *Consider the n-by-n matrix that has zeros everywhere except for 1's on the ascending diagonal:*

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \vdots & 0 & 1 & 0 \\ \vdots & \cdots & . & \cdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Calculate (again show a few steps to indicate your reasoning)

(a) $\quad \mathbf{AA} = \mathbf{I}$ (by direct calculation)

(b) $\quad \mathbf{A^9} = (\mathbf{A^2})^4\mathbf{A} = \mathbf{A}$

(c) $\quad \mathbf{A^{-1}} = \mathbf{A}$ from (a)

(d) $\quad \mathbf{A^T} = \mathbf{A}$ by inspection

(e) $\quad \det \mathbf{A} = \pm 1$ since $\mathbf{A^{-1}} = \mathbf{A}$ and $\det(\mathbf{A^{-1}}) = \mathbf{1}/\det(\mathbf{A})$. But we can go further. $\mathbf{A}$ is obtained by permutations of the rows of the identity matrix $\mathbf{I}$, by swapping the first and last row of $\mathbf{I}$, then its 2nd and next to last rows, etc... Each permutation gives a factor of -1. If $m = n/2$ is the greatest integer less or equal to $n/2$ with n the dimension of the matrix, then $\det(\mathbf{A}) = (\mathbf{-1})^{\mathbf{m}}$.

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5. *Determine whether the following vectors are linearly independent or dependent. If they are L.D., find a linear relation between them.*
(a) $\mathbf{u} = (2, 1, 5), \quad \mathbf{v} = (2, 2, -1), \quad \mathbf{w} = (-2, 0, 3)$.
(b) $\mathbf{u} = (0, 1, 1), \quad \mathbf{v} = (2, 2, -1), \quad \mathbf{w} = (-2, 0, 3)$.
Linear independence means that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0$ if and only if $a = b = c = 0$.

(a) corresponds to the linear system

$$\begin{pmatrix} 2 & 2 & -2 \\ 1 & 2 & 0 \\ 5 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

By Gaussian Elimination: row3 := row3 - 5 row2; row2 := row2 - (row1)/2; Then row3:= row3 + 11 row2

$$\begin{pmatrix} 2 & 2 & -2 \\ 1 & 2 & 0 \\ 5 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & -11 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 14 \end{pmatrix}$$

Therefore the only solution is $a = b = c = 0$ and the vectors are LI.

(b) swapping row1 and row3, then eliminating

$$\begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & 0 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

The matrix is singular so the vectors are linearly dependent. The system yields $b = c$ and $a = b - 3c = -2c$ for any c. Taking $c = 1$ we get $-2\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$.