1. A snowboarder slides (without friction) from (0,0) to $(\pi R, -2R)$. This takes time $T_c = \pi\sqrt{R/g}$ if the track is a cycloid (as we calculated in class).
Key Mathematical concept: velocity = distance/time. If moving along a curve, distance= arclength, so

$$v = \frac{ds}{dt} \tag{1}$$

Key Physical concept: conservation of energy $v^2/2 + gy = Const.$, where y points upward, opposite to gravity. Here, we always start at (0,0) with $v = 0$ so the constant is always 0 and

$$\frac{v^2}{2} = -gy \tag{2}$$

(y points upward so we can only move towards $y < 0$. Math makes sense).
Combining (1) and (2)

$$\frac{ds}{dt} = \sqrt{-2gy} \quad \Longleftrightarrow \quad dt = \frac{ds}{\sqrt{-2gy}} \tag{3}$$

It's now just a matter of cranking it out if you understood arclength and parametrization of curves. Use Maple's `int` and `evalf` if needed but you must first "scale out" R and g with simple substitutions. All times should be proportional to $\sqrt{R/g}$, it's the only physical time in the problem.

(a) Straight line: must be $x = -\pi y/2$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{\pi^2 + 4}\ |dy|/2$ and the time to go from (0,0) to $(\pi R, -2R)$ along that straight line is (integrating (3)):

$$T_l = \int_{-2R}^{0} \frac{\sqrt{\pi^2+4}}{2\sqrt{-2gy}}\, dy = \sqrt{\pi^2+4}\sqrt{\frac{R}{g}} \approx 3.724\sqrt{\frac{R}{g}}. \tag{4}$$

(b) Sinusoid: must be $y = -2R\sin(x/(2R))$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \cos^2(x/(2R))}\,|dx|$ and

$$T_s = \frac{1}{2\sqrt{gR}} \int_0^{\pi R} \frac{\sqrt{1 + \cos^2\frac{x}{2R}}}{\sqrt{\sin\frac{x}{2R}}}\, dx = \sqrt{\frac{R}{g}} \int_0^{\pi/2} \frac{\sqrt{1+\cos^2 u}}{\sqrt{\sin u}}\, du \approx 3.36363\sqrt{\frac{R}{g}} \tag{5}$$

where the last integral is done with Maple's `evalf`.

(b) Parabola: must be $x = \pi y^2/(4R)$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{\pi^2 y^2 + 4R^2}\ |dy|/(2R)$ and

$$T_p = \frac{1}{2R\sqrt{2g}} \int_0^{2R} \frac{\sqrt{4R^2 + \pi^2 y^2}}{\sqrt{y}}\, dy = \sqrt{\frac{R}{g}} \int_0^1 \frac{\sqrt{1+\pi^2 u^2}}{\sqrt{u}}\, du \approx 3.1724\sqrt{\frac{R}{g}} \tag{6}$$

Summary: $T_l > T_s > T_p > T_c$, cycloid is fastest, line is slowest. Try it at Tyrol basin!

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2. See picture p. 661. You may follow the hints there and guess that $r = Ce^{-\theta}$ but need to thorougly show that this is correct. (Briefly, by symmetry the angle between the radial line joining the center to an ant and the ant direction is always $\pi/4$, so trajectory must be an

equiangular spiral (we saw that one before), should show that $r = Ce^{-\theta}$ indeed as this $\pi/4$ equi-angular property.).
More general approach: take Cartesian axes centered at the center of the square. The center does not move. Coordinates of one ant are $(x_1, y_1) \equiv (x, y)$ so the coordinates (x_2, y_2) of the next ant in the counterclockwise direction are rotated by $\pi/2$, that's $(x_2, y_2) = (-y, x)$. Now the ant at (x_1, y_1) moves in the direction of the ant at (x_2, y_2) i.e.

$$\frac{dy}{dx} = -\frac{y_1 - y_2}{x_1 - x_2} = \frac{y - x}{y + x}.$$

This is a *scale-invariant* differential equation and could be solved as seen in 15.1. However it is easier in this case to switch to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$ and

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

Then by the chain rule $dy/dx = (dy/d\theta)/(dx/d\theta)$, so after some simple algebra

$$\frac{dr}{d\theta} = -r \quad \Rightarrow \quad r = Ce^{-\theta}.$$

Finally at $\theta = \pi/4$, $r = L/\sqrt{2}$ so $C = Le^{\pi/4}/\sqrt{2}$. Compute the length by simple application of arclength in polar coordinates integrating from $\theta = \pi/4$ to $\theta = \infty$.

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3. Stewart 10.2 # 70. Obviously some Geometric Series business. If R_i is the radius of a circle and D_i is the distance of its center from a vertex, then $R_i = D_i \sin\pi/6$, true for any circle. For the biggest circle: $R_1 = 1/(2\sqrt{3})$. The size of the next circle toward the vertex is determined from the equality $D_{i+1} + R_{i+1} = D_i - R_i$ (do you see where that comes from?). So $R_{i+1} = R_i/3$ and $A = 11\pi/96 \approx 0.35997$, The area of the triangle is $\sqrt{3}/4 \approx 0.43301$.

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4. Calculate $\sin 31^o$ (sine of 31 degrees) to 6 decimal places using only the 4 basic operations $+, -, *, /$. Obviously some Taylor Series application.
TRAP 1: work with those useless degrees! In radians $31^o \equiv 31\pi/180\ rad = \pi/6 + \pi/180$. Then by **Taylor's Formula:**

$$\sin x = \sin a + \cos a\ (x-a) - \sin a\ \frac{(x-a)^2}{2} - \cos a\ \frac{(x-a)^3}{3!} + \sin a\ \frac{(x-a)^4}{4!} + \cdots + R_n$$

where we know some good things about the remainder R_n (see page 636) namely in this sine case $|R_n| < |x-a|^{n+1}/(n+1)!$.
TRAP 2: Expand about $a = 0$. Smarter to pick $a = \pi/6$, for which basic geometry gives $\sin a = 1/2$ and $\cos a = \sqrt{3}/2$. We should safely get 6 digits if $R_n < 10^{-7}$ i.e. $(\pi/180)^{n+1} < (n+1)!/10^7$, this is true for $n \geq 3$. The $n = 3$ approximation is

$$\sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{180} - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{180}\right)^3 \approx 0.51503807296522$$

which is actually correct to 8 digits. (BTW, what is the limit of the sequence $a_1 = 1.5$, $a_{n+1} = 0.5(a_n + 3/a_n)$? calculate a few terms. See #23 in Sect. 2.10. Could you have competed with the ancient Babylonians?)