1. The integrand is bounded (indeed it is ≤ 1 and $\lim_{x\to 0} \sin x/x = 1$) in a finite interval. The integrand is not formally defined at $x = 0$ but nobody cares about that *single* point as far as "area under the curve" is concerned. Some purists may want to call it improper but it's proper enough for most of us. However if we use the trapezoidal rule, we may have a problem because it will want to use $x = 0$ (see 7.8 formula 4) where the integrand is not defined unless you explicitly define it as equal to 1. The midpoint rule has no problems because it never uses $x = 0$ and the integrand is quite nice as long as $x \neq 0$. Try the midpoint rules with increasing number of rectangles to get an idea of the convergence of the method as the number of rectangles increases. Maple gives

```
evalf(middlesum(sin(x)/x,x=0..Pi,5));          1.857196808
evalf(middlesum(sin(x)/x,x=0..Pi,10));         1.853247529
evalf(middlesum(sin(x)/x,x=0..Pi,20));         1.852264395
evalf(middlesum(sin(x)/x,x=0..Pi,40));         1.852018870
```

The various approximations are clearly converging and we can trust that the first 3 digits are 1.85. If you want to be a little fancier, you can use the known fact (see 7.8) that midpoint converges quadratically, i.e. the error is $O(1/N^2)$ where N is the number of rectangles. In other words, we expect the 10 rectangle approximation, call it M_{10} to be 4 times better than the 5 rectangle approximation, called M_5. We can *extrapolate*, *i.e.* a better estimate is $(4M_{10} - M_5)/3$ as this kills the $1/N^2$ contribution to the error:

$$\frac{4M_{10} - M_5}{3} = 1.851931102.$$

This gives an approximation correct to 5 digits using two approximations that only had 3 correct digits! Cool. Good answer with little work.

2.

$$\int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}\, dx.$$

This is an improper integral, the integrand blows-up at $x = \pm 1$. Trapezoid would not work because it wants to evaluate the integrand at $x = \pm 1$. Midpoint would work but converge poorly (*worse* than the $1/N^2$ advertised in 7.8). Best way is to use the $x = \sin\theta$ substitution (see exam 1 solutions) then

$$\int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}\, dx = \int_{-\pi/2}^{\pi/2} e^{\sin\theta}\, d\theta.$$

That θ integral looks pretty hard. You cannot do it analytically (nobody can) but it looks beautiful to both the trapezoidal and midpoint rules. They will both work *fabulously* well, MUCH better than the $1/N^2$ advertised in 7.8. Amazing what a little substitution can do!

```
evalf(trapezoid(exp(sin(t)),t=-Pi/2..P/2,3));       3.977604561
evalf(trapezoid(exp(sin(t)),t=-Pi/2..Pi/2,6));      3.977463260
```

All the digits of the 6 rectangle approximation are actually correct!

3. $x(x+1)y' - y = x(x+1)$ for $x > 0$ with $y(1) = A$, is a linear first order Diff Eq. The integrating factor is $(x+1)/x$.

$$y(x) = \frac{x}{x+1}(2A + \ln x + x - 1).$$

---

4. $y' = e^{x-y}$ with $y(0) = 1$. First order nonlinear but separable: $e^y dy = e^x dx$ so $e^y = e^x + C$ or $y(x) = \ln(C + e^x)$. Need $y(0) = 1 = \ln(C+1)$ so $C = e - 1$ and $y(x) = \ln(e - 1 + e^x)$.

---

5. $y(x) = 3e^{-x^2} + \int_1^x e^{u^2 - x^2} \cos u^3 \, du$ satisfies what 1st order diff eq?
This is basically the same problem as #2 on TH1 and # 1.d on Exam 1, so by now it is obvious that $y(1) = 3/e$ and $y' + 2xy = \cos x^3$, isn't it?

---

6. Pendulum. Remember that $\dot{x}$ is Newton's notation for dx/dt. Then follow the hints:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \Longrightarrow \dot{\theta}\ddot{\theta} + \frac{g}{L}\dot{\theta}\sin\theta = 0 \Longrightarrow \frac{d}{dt}\frac{(\dot{\theta})^2}{2} - \frac{g}{L}\frac{d}{dt}\cos\theta = 0$$

$$\Rightarrow (\dot{\theta})^2 - \frac{2g}{L}\cos\theta = C \Longrightarrow \dot{\theta} = \pm\sqrt{\frac{2g}{L}}\sqrt{\cos\theta - \cos\theta_0}.$$

after using $\dot{\theta}(0) = 0$ and $\theta(0) = \theta_0$ to determine $C = -(2g/L)\cos\theta_0$. Note that the equation implies $|\theta| \leq |\theta_0|$, the pendulum never goes higher than its initial position. C is the *total energy*, kinetic + potential. Energy is conserved.
The first order equation for θ is nonlinear but separable

$$\frac{\pm d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{2g}{L}}\, dt.$$

Integrating both sides from $t = 0$ to T, the time it takes to go from $\theta = \theta_0$ to $\theta = 0$, gives

$$\int_0^T dt = T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \qquad (T)$$

This is an improper integral as the integrand blows up at $\theta = \pm\theta_0$ but it converges because $\cos\theta \approx \cos\theta_0 - \sin\theta_0(\theta - \theta_0)$ (formula (5) in 2.9), so $\sqrt{\cos\theta - \cos\theta_0} \approx \sqrt{\sin\theta_0}\sqrt{\theta_0 - \theta}$ for θ near θ_0 (*i.e.* $|\theta - \theta_0| \leq \epsilon \ll \theta_0$) and

$$\int_{\theta_0 - \epsilon}^{\theta_0} \frac{d\theta}{\sqrt{\theta_0 - \theta}} = \int_0^{\epsilon} \frac{dx}{\sqrt{x}} = 2\sqrt{\epsilon} \quad < \infty.$$

(Let $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin x$ to get the integral (T) in the form given in 7.8, #36.)