Problem 1

We wish to integrate numerically the following integral: $\int_0^\pi \frac{\sin x}{x}dx$, using the trapezoidal and midpoint methods. First note that that even though the integral looks improper, $\lim_{x\to 0}\frac{\sin x}{x} = 1$; hence it exists. The graph looks fairly smooth and "well-behaved"; therefore we should not expect any problems. We wish to know the answer to 3 decimal digits, then we can use the estimates for the error bounds for the Trapezoidal and Midpoint Rules. Using Theorem 5 on page 460, we respectively get:

$$|E_T| \leq \frac{K\pi^3}{12n^2} \tag{1}$$
$$|E_M| \leq \frac{K\pi^3}{24n^2} \tag{2}$$

where n is the number of intervals used in the method and K is an upper bound for the absolute value of the second derivative of $\frac{\sin x}{x}$. Differentiating twice we get

$$\frac{d^2}{dx^2}(\frac{\sin x}{x}) = -\frac{2\cos x}{x^2} + 2\frac{\sin x}{x^3} - \frac{\sin x}{x}$$

Using the following inequalities for $x \in [0, \pi]$:

$$\begin{aligned} 1 \geq \cos x &\geq 1 - \frac{x^2}{2} \\ x \geq \sin x &\geq x - \frac{x^3}{x}, \end{aligned}$$

we obtain $-\frac{1}{3} \leq -\frac{2\cos x}{x^2} + 2\frac{\sin x}{x^3} \leq 1$ Hence, $|\frac{d^2}{dx^2}(\frac{\sin x}{x})| \leq |2(\frac{1}{2} - \frac{1}{x^2}) + \frac{2x}{x^3} - |\frac{\sin x}{x}|| \leq |2(\frac{1}{2} - \frac{1}{x^2}) + \frac{2x}{x^3}| + |\frac{\sin x}{x}| = 1 + |\frac{\sin x}{x}| \leq 2$.

Hence, we can use $K = 2$ in the formulas 1 and 2. Since we want at least 3 correct deciamal digits we need, $|E_T| \leq 10^{-4}$, for the Trapezoidal Rule and $|E_M| \leq 10^{-4}$, for the Midpoint one. The solutions to these inequalities are, respectively: $n_T \geq 228$, and $n_M \geq 161$ for the Trapezoidal and Midpoint Rules. These values for n_T and n_M ensure that we will get the answer correct to at least 3 decimal places. When we apply the formula we get the following approximations:

$$\begin{aligned} \int_0^\pi \frac{\sin x}{x}dx &\approx 1.851932. \text{ for the Trapezoidal Rule;} \\ \text{and } \int_0^\pi \frac{\sin x}{x}dx &\approx 1.851942 \text{ for the Midpoint Rule.} \end{aligned}$$

In both of these methods we've defined $\frac{\sin x}{x}\Big|_{x=0} = 1$, because $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Comparing these number to the value 1.851937, given by Mathematica, with the Trapezoidal rule we got 5 correct decimal places; with the Midpoint Rule we got 4 correct decimal places. Also, the Trapezoidal rule underestimates the "true" value by about -5×10^{-5}; the Midpoint rule overestimates it by 5×10^{-5}. So the two methods are essentially the same. Also note that the formulas 1 and 2 are only **upper** bounds for the error. The **actual** error may well be smaller than the given bound. In fact, in most of the cases, the actual error is, indeed, less than the predicted. Furthermore, more often than not, it is difficult to estimate K. So, what is usually done in practice is to run successive approximations by increasing the value of n, and stop when you get the difference of two successive estimates less than the desired accuracy. As you will see, in Problem 2, this method is much more efficient than simply applying the formulas from the book.

Problem 2 We saw in the exam that

$$\int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\sin t} dt \tag{3}$$

Looking at the graph of $f(t) = e^{\sin t}$, we again see no "irregularities"; hence a good approximation can be given by the Trapezoidal Rule. Using $n = 100$, we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt \approx 3.97746326050642263725660983266469716408$$

Using formula 1 from problem 1, we get the following upper bound for the error: $|E_T| \le 0.00140473$, which means that the estimated value of the integral is correct to at least 2 decimal places. On the other hand, Mathematica estimates the actual error to be about 4.96×10^{-435}, which is amazing!. So, the Trapezoidal rule gives an excellent approximation to the integral. In fact it gives a correct answer to 15 decimal places using only $n = 5$. This shows that one should use given formulas *wisely* . As you can see in this problem blind application of the error bound formula leads to a *very inefficient* method. This argues the case of the alternative method, suggested in Problem 1—successive approximation. Once again, the given formulas are simply guides. One should think how they apply to the particular problem. In this case, even though the given bound for $|E_T|$, even though correct, is vastly inflated. So, we need a better theory for bounds in this case.

Finally, the result $|E_T| \le 0.00140473$ is obtained by using formula (1) and noting that $f''(t) = e^{\sin x}\cos(x^2) - e^{\sin x}\sin x$. Because both the *cosine* and the *sine* functions are bounded by 1, and the exponential is an increasing function, we get $|f''(t)| \le 2e$.

Problem 3 We wish to solve the differential equation

$$\begin{aligned} x(x+1)y' - y &= x(x+1) \text{ for } x > 0 \\ \text{with } y(1) &= A \end{aligned} \tag{4}$$

This is a linear differential equation. To convert it into the standard form $y' + P(x)y = Q(x)$ we divide both sides of the equation by $x(x+1)$. Then $P(x) = -\frac{1}{x(x+1)}$ and $Q(x) = 1$ and the integrating factor is $e^{\int P(x)dx} = \frac{x+1}{x}$. Here we also use the fact that $x > 0$. So $\frac{d}{dx}(y\frac{x+1}{x}) = \frac{x+1}{x}$. The solution to the equation then becomes

$$y\frac{x+1}{x} = \int_1^x \frac{(t+1)dt}{t} + 2A.$$

The formula above uses the initial condition given. Carrying out the integration we obtain the final answer to (4) as

$$y = \frac{x}{x+1}\left(x - 1 + \ln x + 2A\right)$$

Problem 4 Now we wish to solve the differential equation

$$\begin{aligned} y' &= e^{x-y} \\ \text{with } y(0) &= 1 \end{aligned} \tag{5}$$

Equation (5) is a separable one and it separates into $e^y dy = e^x dx$. Integrating both sides we get the general solution to be $e^y = e^x + C$. Using some algebra and the initial condition we get $C = e - 1$, and thus the solution to the equation is

$$y = \ln(e^x + e - 1)$$

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Problem 5 We are given the following differential equation:

$$y(x) = 3e^{-x^2} + \int_1^x e^{u^2 - x^2} \cos(u^3) du \tag{6}$$

To find a differential equation to which (6) is a solution we must differentiate both sides of the equation. It is useful, however, to write it first as: $y(x) = 3e^{-x^2} + e^{-x^2} \int_1^x e^{u^2} \cos(u^3) du$. Now we can differentiate easily. Using the Fundamental Theorem of Calculus *and* the product rule we obtain $y'(x) = -6xe^{-x^2} + 2xe^{-x^2} \int_1^x e^{u^2} \cos(u^3) du + \cos(x^3)$. Remembering that we started with 6 we rearrange the result into

$$y'(x) + 2xy = \cos(x^3).$$

That is the differential equation we were looking for. The initial condition is given by $y(1) = 3e^{-1}$.

Problem 6 Given is the pendulum equation:

$$\begin{aligned} \ddot{\theta} + \frac{g}{L} \sin\theta &= 0 \qquad (7) \\ \text{with initial conditions: } \dot{\theta}(0) &= 0 \\ \text{and } \theta(0) &= \theta_0 \end{aligned}$$

For the sake of notation let $K = \frac{g}{L}$.

Multiply equation (7) by $2\dot{\theta}$ and you get $2\dot{\theta}\ddot{\theta} + 2K\dot{\theta}\sin\theta = 0$. Recall that by the chain rule we have $d(\dot{\theta})^2/dt = 2\dot{\theta}\ddot{\theta}$ and that $-d\cos\theta/dt = \dot{\theta}\sin\theta$. Hence, we have reduced the order of the original equation; the new equation, after integrating, becomes

$$\begin{aligned} \left(\frac{d\theta}{dt}\right)^2 &= 2K\cos\theta + C \\ \text{or } \frac{d\theta}{dt} &= \sqrt{2K\cos\theta + C} \qquad (8) \end{aligned}$$

Equation (8) is the first order equation for θ that we get from equation (7). We can get the value of C from the initial conditions. They yield $C = -2K\cos\theta_0$. Now equation (8) reduces to the following initial value problem:

$$\begin{aligned} \frac{d\theta}{dt} &= \sqrt{2K(\cos\theta - \cos(\theta_0))} \qquad (9) \\ \text{with initial condition: } \theta(0) &= \theta_0 \end{aligned}$$

This is a separable equation. It separates into $\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{2K}$. Given the initial condition its solution is

$$\int_{\theta_0}^{\theta} \frac{dx}{\sqrt{\cos x - \cos\theta_0}} = t\sqrt{\frac{2g}{L}} \tag{10}$$

The only remaining problem is that the integral in the solution (10) is improper. To show that the integral is convergent, we can prove that $\cos x - \cos\alpha \geq \frac{\alpha^2 - x^2}{2}$. Let $f(x) = \cos x - \cos\alpha - \frac{\alpha^2-x^2}{2}$. Then $f'(x) = x - \sin x \geq 0$. Hence, $f(x)$ is an increasing function of$x$. Thus, when $x \geq \alpha$, $f(x) \geq f(\alpha) = 0$. This inequality shows that

$$\frac{1}{\sqrt{\cos x - \cos\theta_0}} \leq \sqrt{\frac{2}{x^2 - \theta_0^2}} \leq \frac{1}{\sqrt{\theta_0}} \frac{1}{\sqrt{x - \theta_0}}.$$

The last inequality stems from the fact that $x^2 - \theta_0^2 = (x + \theta_0)(x - \theta_0)$, and that $x \geq \theta_0$. Since, the integral $\int_{\theta_0}^{\theta} \frac{dx}{\sqrt{x-\theta_0}}$ exists, we claim that the integral in (10) is convergent. So, in that respect we are all right.

Finally, the time it takes for the pendulum to from its initial position at θ_0 to the vertical position $\theta = 0$ is given by the equation:

$$T = \frac{L}{2g}\left(\int_{\theta_0}^{0} \frac{dx}{\sqrt{\cos x - \cos\theta_0}}\right)$$

And this completes the solution to problem 6.

As a further comment, if you run the integral in (10) in Mathematica, it gives you the following formula:

$$\int_{\theta_0}^{\theta} \frac{dx}{\sqrt{\cos x - \cos\theta_0}} = \frac{2}{\sqrt{1 - \cos\theta_0}} \int_{\frac{\theta_0}{2}}^{\frac{\theta}{2}} \frac{dx}{\sqrt{1 - \frac{1}{1-\cos\theta_0}\sin^2 x}}. \tag{11}$$

The second integral in (11) is called an elliptic function. It can be easily verified by differentiation that this is the same solutions as 10. Then, the period T of the pendulum can also be expressed as:

$$T = \frac{L}{g} \int_{\frac{\theta_0}{2}}^{0} \frac{dx}{\sqrt{1 - \cos\theta_0 - \sin^2 x}}$$