1. [20 pts] *Evaluate the following limits, showing the main steps in your reasoning.*

$$\lim_{x\to -a} \frac{x^2+a^2}{x^3-a^3} \tag{1}$$

No problem with this one. As $x \to -a$ the numerator $\to 2a^2$ and the denominator $\to -2a^3$ so the limit is $-1/a$.

$$\lim_{\epsilon\to 0} \frac{\sqrt{4+\epsilon^2}-2}{\epsilon^2} \tag{2}$$

Remember derivative of $\sqrt{x}$ (example 5, p. 35) and the algebraic formula $a^2-b^2=(a-b)(a+b)$. Think $a=\sqrt{4+\epsilon^2}$, $b=2$. So we multiply top and bottom by $a+b$ *i.e.* $\sqrt{4+\epsilon^2}+2$ and get

$$\frac{\sqrt{4+\epsilon^2}-2}{\epsilon^2} = \frac{\epsilon^2}{\epsilon^2\left(\sqrt{4+\epsilon^2}+2\right)} = \frac{1}{\sqrt{4+\epsilon^2}+2} \to \frac{1}{4}.$$

This limit is in fact the definition of the derivative of $\sqrt{x}$ at $x=4$ (with ϵ^2 in place of our usual h).

$$\lim_{t\to +\infty} \frac{\sqrt{5+t^2}-\sqrt{5}}{t} \tag{3}$$

As $t\to\infty$ the t^2 becomes really big and those tiny 5's cannot keep up so the ratio is $\approx \sqrt{t^2}/t = 1$ so the limit is 1. You can do this more cleanly by taking t^2 out of the $\sqrt{\ }$

$$\frac{\sqrt{5+t^2}-\sqrt{5}}{t} = \frac{t\sqrt{1+5/t^2}-\sqrt{5}}{t} = \sqrt{1+\frac{5}{t^2}} - \frac{\sqrt{5}}{t} \to 1$$

as $t\to+\infty$.

$$\lim_{s\to 1} \frac{s^2-4s+3}{s^2+2s-3} \tag{4}$$

Top and bottom both go to zero so it's not trivial. However that also means $s=1$ is a root of both s^2-4s+3 and s^2+2s-3, in other words $s^2-4s+3=(s-1)(s-a)$ and $s^2+2s-3=(s-1)(s-b)$. With a tiny bit of work, you'll find $a=3$ and $b=-3$. So

$$\lim_{s\to 1} \frac{s^2-4s+3}{s^2+2s-3} = \lim_{s\to 1} \frac{s-3}{s+3} = -\frac{1}{2}.$$

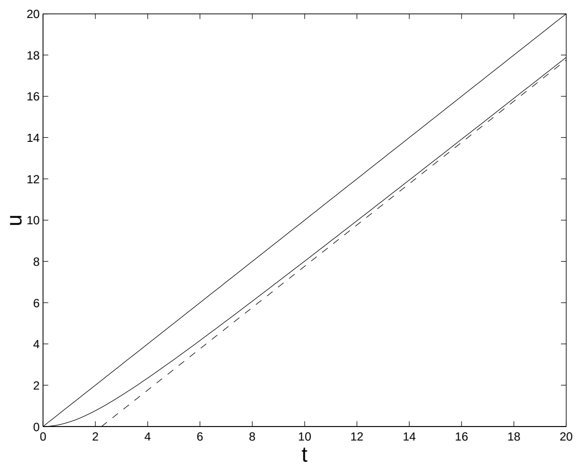
In relation with limit (3), sketch the curves $u=t$ and $u=\sqrt{5+t^2}-\sqrt{5}$ for $t\ge 0$. The sketch should illustrate the result you obtained for the limit (3). Specify the exact values of t at which the curves intersect, if they do ($\sqrt{2}$ is exact but 1.414 is not).

The curves cross when $t=\sqrt{5+t^2}-\sqrt{5}$ *i.e.* $(t+\sqrt{5})^2 = 5+t^2$ or $2\sqrt{5}t=0$, so $t=0$, no other intersection. We know that the limit of the ratio is 1, but that doesn't necessarily mean $\sqrt{5+t^2}-\sqrt{5}\approx t$ for large t. To see that, let's look at the difference between those two functions: $d(t)\equiv \sqrt{5+t^2}-\sqrt{5}-t$. What does $d(t)$ tend to as $t\to\infty$? Taking t out of the $\sqrt{\ }$ as before we get $d(t)\to -\sqrt{5}$ as $t\to\infty$. So the function $\sqrt{5+t^2}-\sqrt{5}\approx t-\sqrt{5}$ for large t. A bit tricky!

Anyway, the curve $u=\sqrt{5+t^2}-\sqrt{5}$ approaches the curve $u=t-\sqrt{5}$ as $t\to\infty$. We also know that it intersects the curve $u=t$ only at $u=t=0$, so it must lie below the curve $u=t$. What else do we need to sketch the curves? the slope at $t=0$ helps:

$$\frac{d}{dt}\left(\sqrt{5+t^2}-\sqrt{5}\right) = \frac{t}{\sqrt{5+t^2}} \quad = 0 \ \text{ at } \ t=0.$$

Here's a sketch. The dash line is $u=t-\sqrt{5}$.



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2. [10pts] *What is the range of x such that* $\dfrac{1}{(x+4)^2} > 100$?

We need $(x+4)^2 < 10^{-2}$ or $|x+4| < 10^{-1} = 0.1$. This means that the distance between x and -4 must be less than 0.1. In other words: $-4.1 < x < -3.9$.

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3. [20pts] *Evaluate dy/dx for each of the following*

$$y = \frac{\sqrt{2-x}}{3} \Rightarrow \frac{dy}{dx} = -\frac{1}{6\sqrt{2-x}} \tag{5}$$

$$y = \frac{x}{x^2-1} \Rightarrow \frac{dy}{dx} = \frac{1(x^2-1) - x(2x)}{(x^2-1)^2} == -\frac{x^2+1}{(x^2-1)^2} \tag{6}$$

$$y = (x + x^{-1})^2 \Rightarrow \frac{dy}{dx} = 2(x + x^{-1})(1 - x^{-2}) \tag{7}$$

$$y = b + \sqrt{R^2 - (x-a)^2} \Rightarrow \frac{dy}{dx} = \frac{-(x-a)}{\sqrt{R^2-(x-a)^2}} \tag{8}$$

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4. [10pts] *Two variables, s and t, are related by the equation $t = s^3 - s^2 + s$. Find an integer value of s such that $t = 21$. What is the rate of change of s at $t = 21$?*

$s = 3 \Rightarrow t = 21$.

(1) By CHAIN RULE: Take d/dt of both sides, thinking $s = s(t)$. Then $1 = 3s^2s' - 2ss' + s'$, where $s' \equiv ds/dt$. Solving for s' gives $s' = 1/(3s^2 - 2s + 1)$. At $t = 21$ we have $s = 3$, so $ds/dt = 1/22$.

(2) By INVERSE FUNCTIONS:

$$\frac{ds}{dt}(t=21) = \frac{1}{\frac{dt}{ds}(s=3)} = \frac{1}{3s^2 - 2s + 1}\Big|_{s=3} = \frac{1}{22}.$$

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5. [10pts] *Consider the function*

$$f(x) = \begin{cases} x^2 + bx + c, & \text{for} \quad x < 0 \\ x + 2, & \text{for} \quad x \geq 0 \end{cases}$$

Find the constants b and c such that $f(x)$ and $f'(x)$ are continuous everywhere then sketch $f(x)$.

A function $f(x)$ is continuous at a point a if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$. A function is continuous on a domain, if is is continuous at all points of that domain. Here we know $f(x)$ and $f'(x)$ are continuous for $x < 0$ and $x > 0$ (we know that because they are simple functions with simple derivatives for which we can easily show continuity). The problem is clearly at $x = 0$. But we find easily that

$$\lim_{x\to 0^+} f(x) = 2, \quad \lim_{x\to 0^+} f'(x) = 1,$$

and

$$\lim_{x\to 0^-} f(x) = c, \quad \lim_{x\to 0^-} f'(x) = b.$$

So for continuity of the function and its derivative, we need $b = 1$, $c = 2$.

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6. [10pts] *The equation $(x^2 + y^2)(x^2 + y^2 + x) = 4xy^2$ defines a famous curve. What is its slope at $x = 1/2$, $y = \sqrt{3}/2$?*

By implicit differentiation, thinking $y = y(x)$, we take d/dx of both sides to get, with $y' \equiv dy/dx$:

$$(2x + 2yy')(x^2 + y^2 + x) + (x^2 + y^2)(2x + 2yy' + 1) = 4y^2 + 8xyy'.$$

Now we solve for y' using straightforward algebra. Focus on y' and no need to do more algebra than necessary, we don't need to expand all out. Regroup the y''s on the left:

$$2x(x^2 + y^2 + x) + (x^2 + y^2)(2x + 1) + \Big[2y(x^2 + y^2 + x) + 2y(x^2 + y^2)\Big]\, y' = 4y^2 + 8xyy',$$

Then put all the y' on one side, cleaning up easy algebra along the way:

$$2x(x^2 + y^2 + x) + (x^2 + y^2)(2x + 1) - 4y^2 = \Big[8xy - 4y(x^2 + y^2) - 2xy\Big]\, y',$$

so finally

$$y' = \frac{2x(x^2 + y^2 + x) + (x^2 + y^2)(2x + 1) - 4y^2}{6xy - 4y(x^2 + y^2)}.$$

Now we just let $x = 1/2$, $y = \sqrt{3}/2$

$$y' = \frac{(1/4 + 3/4 + 1/2) + (1/4 + 3/4))(1 + 1) - 3}{3\sqrt{3}/2 - 4\sqrt{3}/2} = -1/\sqrt{3}.$$

If you wonder what the curve looks like, it's easier to use *polar coordinates* $x = r\cos\theta$ and $y = r\sin\theta$, as we did in class today. Substituting in the equation, we obtained $r = (4\sin^2\theta - 1)\cos\theta$ and this yields the *parametric representation* of the curve:

$$x = (4\sin^2\theta - 1)\cos^2\theta,$$

$$y = (4\sin^2\theta - 1)\frac{\sin(2\theta)}{2}.$$

This gives one and only one x and y for any θ. This representation shows that the curve is closed and inside the circle of radius 3 centered at the origin. It's a pretty curve, plot it out.

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7. [10pts] Your TA will tell you. The sketch looks like an upright parabola crossing the $y = 0$ axis at the two x's corresponding to the local maximum and minimum of f(x).

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8. [10pts] *Find the derivative of the function*

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

[Hint: we have not covered derivatives of trig functions yet but you do not need them here. You need the fundamental definition of the derivative and you can squeeze the answer. You may need a sandwich.]

$$f'(0) \equiv \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} h \sin(1/h).$$

We've dealt with this limit in class using the sandwich (or squeeze) theorem:

$$-1 \le \sin(1/h) \le 1,$$

so

$$-|h| \le |h| \sin(1/h) \le |h|$$

and given that $|h| \to 0$ as $h \to 0$, the sandwich theorem gives that $|h| \sin(1/h) = \pm h \sin(1/h) \to 0$. So

$$f'(0) = \lim_{h\to 0} h \sin(1/h) = 0.$$