1. [50 pts] Calculate the following expressions. Show your work but NO PARTIAL CREDIT. Full credit only for correct answer with correct derivation.

(1) $\lim_{a\to\infty}\int_0^a \frac{dx}{1+x^2} = \lim_{a\to\infty}[\arctan x]_0^a = \lim_{a\to\infty}\arctan a = \pi/2.$

(2) $\lim_{\theta\to 0^+}\sqrt{\theta}\ln\theta = \lim_{\theta\to 0^+}\frac{\ln\theta}{\theta^{-1/2}} =$ "0/0," so we can use l'Hospital: $= \lim_{\theta\to 0^+}\frac{1/\theta}{(-1/2)\,\theta^{-3/2}} = \lim_{\theta\to 0^+} -2\theta^{1/2} = 0.$

(3) $\int_{-3}^{2}\frac{dx}{\sqrt{17-x^2}} = \int_{-3}^{2}\frac{dx}{\sqrt{17}\sqrt{1-x^2/17}}$ now let $u = x/\sqrt{17}$ and the integral $= \int_{-3/\sqrt{17}}^{2/\sqrt{17}}\frac{du}{\sqrt{1-u^2}} = [\arcsin u]_{-3/\sqrt{17}}^{2/\sqrt{17}} = \arcsin(2/\sqrt{17}) + \arcsin(3/\sqrt{17}).$

(4) $\int_2^1 \frac{\ln x}{x}dx = -\int_1^2 d((\ln x)^2/2) = -\left[(\ln x)^2/2\right]_1^2 = -0.5(\ln 2)^2.$

(5) $\frac{dx^x}{dx} = \frac{d}{dx}\exp(x\ln x) = \exp(x\ln x)(\ln x + 1) \equiv x^x(\ln x + 1).$

(6) $\arccos(\cos(7\pi/4))$ Make a sketch of $y = \cos x$ to see it. $\cos 7\pi/4 = \cos(7\pi/4 - 2\pi) = \cos(-\pi/4) = \cos\pi/4$. By definition, arccos has a range of $[0,\pi]$ so $\arccos(\cos(7\pi/4)) = \pi/4$.

(7) $\cos(\arcsin(3))$ Does not make sense. There is no (real) arc with a sine of 3.

(8) $\sin(\arccos(1/3))$. Let $\alpha \equiv \arccos(1/3)$, then $\cos\alpha = 1/3$ and by Pythagoras $\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{8/9} = \pm 2\sqrt{2}/3$. Which sign is it, $\pm$? The arccos is an angle in $[0,\pi]$ but such angles have positive sines (draw unit circle), so we need to select the + sign.

(9) $\ln(x^3 e^{-x^2}) = 3\ln x - x^2.$

(10) $\lim_{x\to+\infty}\frac{x^a}{a^x}$, $\forall a > 0$, This limit is $= +\infty$ if $a \le 1$ and $=$ "$+\infty/+\infty$" if $a > 1$. We can use l'Hospital for that last case: limit $= \lim_{x\to+\infty}\frac{a x^{a-1}}{a^x \ln a} = 0$ if $a < 2$ otherwise $=$ "∞/∞", so use l'Hospital again to find that the limit is zero if $a < 3$ otherwise ∞/∞, etc... So the limit is zero if $a > 1$

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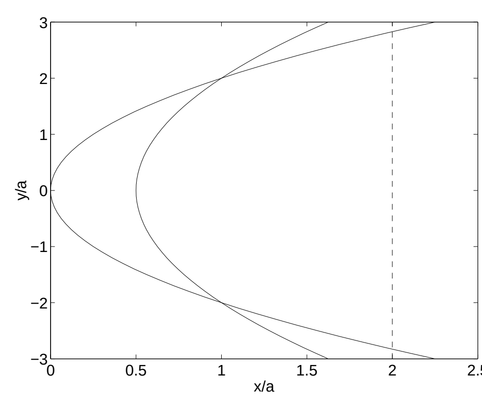
2. [10pts] Calculate (a) the area between the curves $y^2 = 4ax$ and $y^2 = 8ax - 4a^2$, (b) the volume generated by the rotation of that area about the $x = 2a$ axis.

Here's a sketch of the two curves in non-dimensional variables x/a, y/a:



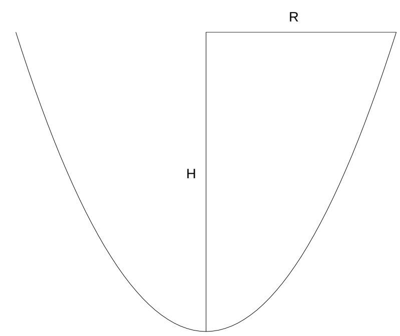
The curves intersect at $y^2 = 4ax = 8ax - 4a^2 \Longrightarrow x = a, y = 2a$.

(a) Enclosed area $= \int_{-2a}^{2a}(x_r(y) - x_l(y))\,dy$ where $x_r(y)$ is the rightmost curve and $x_l(y)$ the leftmost, so $x_r = (y^2+4a^2)/(8a)$ and $x_l(y) = y^2/(4a)$. The area is $= \int_{-2a}^{2a}(a/2 - y^2/(8a))\,dy = 4a^2/3$.

(b) Volume by washers: $V = \int_{-2a}^{2a}\pi(r_0^2 - r_i^2)dy$ where $r_0 = 2a - x_l = 2a - y^2/(4a)$ is the outer washer radius and $r_i = 2a - x_r = 2a - (y^2+4a^2)/(8a)$ is the inner washer radius. There is only straighforward algebra left to do.

3. [10pts] A bowl in the shape of a paraboloid is filled with water at the constant rate of Q $\mathrm{m}^3/\mathrm{sec}$. How fast is the height of water in the bowl rising?

$y = x^2$ is a parabola but what do x and y mean in that equation? If x is a length then $y = x^2$ is a length squared! (area). We encountered this issue in lecture on Fri Nov 16, 2001. A paraboloid of circular base of radius R and of height H is described by the equation $y = Hx^2/R^2$ which is dimensionally correct, x is a length and y is a length. .



If the paraboloid is filled with water up to level $h \neq H$ then the volume of water (by slices) is $V = \int_0^h \pi x^2 dy$, where y is the height of a slice, dy its thickness and x its radius. Now $y = Hx^2/R^2$ so $V = \int_0^h \pi R^2 y/H dy = \pi R^2 h^2/(2H)$ which is indeed a volume.
Then

$$\frac{dV}{dt} = \pi R^2 \frac{h}{H}\frac{dh}{dt} = Q$$

or

$$\frac{dh}{dt} = \frac{QH}{\pi R^2 h}.$$

The units on the right-hand side are indeed m/sec. We can solve that differential equation for $h(t)$: $hdh = Cdt \longrightarrow h^2 = 2Ct + D$ where $C = QH/(\pi R^2)$ and D is an arbitrary constant. If we take $t = 0$ to be the time when there was no water in the bowl, then $D = 0$ and $h = \sqrt{2QHt/(\pi R^2)}$.

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4. [10pts] A radioactive substance disintegrates at a rate proportional to the amount present. If the rate constant is 1 percent per day, how long will it take for the amount to have reduced by half?

$$\frac{dS}{dt} = -0.01S \longrightarrow S(t) = S_0 e^{-0.01t}$$

where S_0 is the initial amount and t is measured in days. $S/S_0 = 1/2 = \exp(-0.01t_*) \longrightarrow \ln(1/2) = -0.01t_*$ or $t_* = 100\ln 2$, about 69.3 days.

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5. [10pts] Suppose you borrow A_0 euros at the rate of r (**%/year**) and interest is compounded continuously. (a) What is the effective *Annual percentage rate*? (b) If you pay back the money *continuously* at the constant rate of p (**euros/year**), what is the differential equation that determines the amount of money owed at time t? (c) Solve that equation.

$$\frac{dA}{dt} = rA - p \longrightarrow \frac{dA}{rA - p} = dt \longrightarrow \ln(rA - p) = rt + C.$$

We get the constant in terms of the initial amount borrowed by evaluating this at $t = 0$, yielding $C = \ln(rA_0 - p)$. Then, taking the exp of both sides:

$$\frac{rA - p}{rA_0 - p} = e^{rt} \longrightarrow A(t) = \frac{p}{r} + (A_0 - \frac{p}{r})e^{rt}.$$

Note that if $A_0 - p/r > 0$ the loan will never be paid back.

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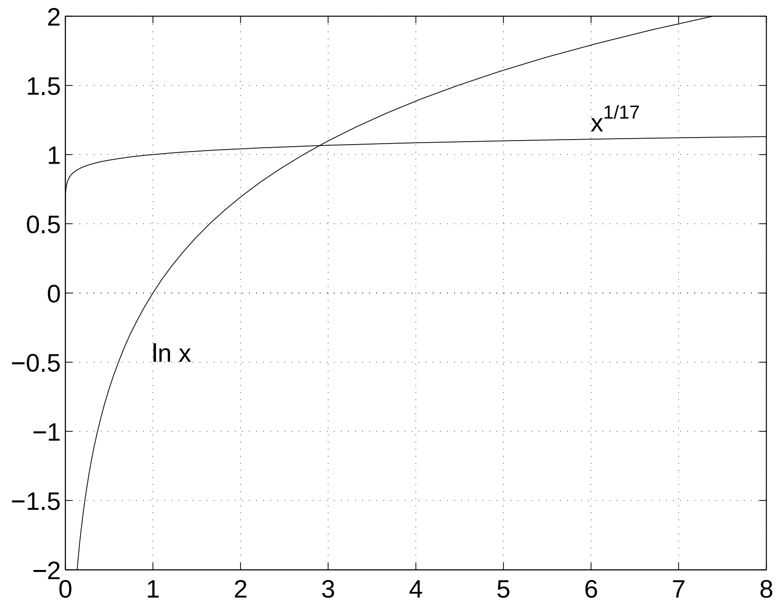
6. [10 pts] Sketch $y = x^{1/17}$ and $y = \ln x$ on the same plot. Calculate (a) $\lim_{x \to 0^+} x^{1/17} \ln x$, (b) $\lim_{x \to +\infty} x^{-1/17} \ln x$. Find a number $M > 0$ such that $x^{1/17} > \ln x$ for all $x > M$.



(a) $\lim_{x \to 0^+} x^{1/m} \ln x = \lim_{x \to 0^+} \dfrac{\ln x}{x^{-1/m}} =$ "0/0". Use l'Hospital: $= \lim_{x \to 0^+} \dfrac{1/x}{(-1/m) x^{-1/m-1}} = \lim_{x \to 0^+} -m x^{1/m} = 0$, $\forall m > 0$.

(b) $\lim_{x \to +\infty} x^{-1/17} \ln x = \lim_{x \to +\infty} \dfrac{\ln x}{x^{1/17}} =$ "∞/∞". Can use l'Hospital again: $\longrightarrow \lim_{x \to +\infty} 17 \frac{1/x}{x^{1/17-1}} = \lim_{x \to +\infty} \frac{17}{x^{1/17}} = 0$. SO $x^{1/17}$ eventually beats the log. It sure does not look like that on the plot!

(c) Let's compare $x^{1/17}$ and $\ln x$ further. How big should x be before $x^{1/17}$ become bigger than $\ln x$? x will have to be really, really big, so maybe we should look at its ln, as the log of a big number is much smaller than that number (log increases slowly). Let $y = \ln x$ then

$$x^{1/17} \equiv e^{(\ln x)/17} > \ln x \Longrightarrow \exp(y/17) > y.$$

That's more manageable. With a bit of fiddling (or Newton's method) you can find that y must be bigger than 72.9189179, meaning that x must be bigger than $e^{72.9189179} \approx 4.6589\, 10^{31}$. A pretty big number indeed.