1. [10 pts] Calculate the following expressions. Briefly explain your reasoning.

(1) $\lim_{x\to 0} \frac{\cos x - x}{x^2} = \text{“}1/0\text{''} = +\infty$.

(2) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x\sin x}\right) = \lim_{x\to 0} \left(\frac{\sin x - x}{x^2 \sin x}\right)$. Can be calculated by l'Hospital three times (!) or using Taylor's formula for $\sin x = x - x^3/6 + O(x^5)$ as $x \to 0$:

$$\lim_{x\to 0} \frac{\sin x - x}{x^2 \sin x} = \lim_{x\to 0} \frac{[x - x^3/6 + O(x^5)] - x}{x^3 + O(x^5)} = \lim_{x\to 0} \frac{-x^3/6 + O(x^5)}{x^3 + O(x^5)} = \lim_{x\to 0} \frac{-1/6 + O(x^2)}{1 + O(x^2)} = -1/6.$$

---

2. Evaluate the following:

(1) $\int \frac{x+1}{\sqrt{x}} dx = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{3/2} + 2\sqrt{x}.$

(2) $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx$ Let $u = \cos x$ so $du = -\sin x dx$ and

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx = \int_{1/\sqrt{2}}^1 du/u^2 = [-1/u]_{1/\sqrt{2}}^1 = \sqrt{2} - 1.$$

(3) $\int_1^x \cos(xu)\, du = \left[\frac{\sin(xu)}{x}\right]_{u=1}^{u=x} = \frac{\sin x^2}{x} - \frac{\sin x}{x}.$

(4) $\frac{d}{dx}\left(\int_1^{x^2} \cos t^2 dt\right) = \frac{d}{du}\left(\int_1^u \cos t^2 dt\right) \frac{du}{dx} = 2x\cos x^4$, where $u = x^2$.

(5) $\lim_{n\to\infty} \frac{1 + 2^5 + 3^5 + \cdots + n^5}{n^6} \equiv \int_0^1 x^5 dx = 1/6.$

(6) $\frac{d}{dt}\left(\int_1^2 \frac{\sin t}{\sqrt{1+t^2}} dt\right) = \frac{d}{dt} Constant = 0.$

---

3. [10pts] A spherical iron ball is coated with ice of uniform thickness. If the ice melts uniformly at a rate proportional to its surface area (*i.e.* at a rate equal to C times the surface area where C is a constant with units of velocity), how long will it take until the ice has disappeared? Express your answer in terms of the parameters of the problem: the size of the ball, the initial ice thickness, the constant C, etc...

Let R be the radius of the ball, $S(t)$ the ice thickness, then

$$\frac{4}{3}\pi \frac{d}{dt}\left((R+S)^3 - R^3\right) = -C4\pi(R+S)^2,$$

therefore

$$\frac{dS}{dt} = -C, \quad \Longrightarrow S(t) = S_0 - Ct$$

where $S_0 \equiv S(0)$ is the initial thickness. The ice is gone when $S = 0$ at $t = S_0/C$.

---

4. [10pts] A window is in the form of a rectangle surmounted by a semicircle. If the rectangle is of clear glass while the semicircle is of colored glass which transmits only half as much light per square foot as clear glass does, and the total perimeter is fixed, find the proportions of the window that will admit the most light.

Let $2x$ be the base the rectangle and y its height. Then the rectangle area is $A_r = 2xy$, the semicircle area is $A_s = \pi x^2/2$. The total amount of light that goes through is proportional to

$A = A_r + A_s/2$. So we want to maximize A subject to fixed perimeter $p_0 = 2x + 2y + \pi x$, which requires $y = (p_0 - \pi x - 2x)/2$. So the function we want to maximize is

$$A(x) = 2xy + \pi x^2/4 = p_0 x - (\pi + 2)x^2 + \frac{\pi}{4}x^2 = p_0 x - \frac{3\pi + 8}{4}x^2.$$

This is an upside-down parabola and its minimum is achieved at $x = 2p_0/(3\pi + 8)$, and $y = (\pi + 4)p_0/(6\pi + 16)$.

---

5. [10pts] Sketch the curve $y = x/(x-1)^3$. Identify all extrema and inflection points.
$y \approx -x$ near $x = 0$, $y \approx 1/x^2$ for large $|x|$, singularity at $x = 1$ and

$$\lim_{x \to 1^-} y(x) = -\infty, \quad \lim_{x \to 1^+} y(x) = +\infty.$$

$$y' = \frac{-2x - 1}{(x-1)^4}, \quad y'' = \frac{6x + 6}{(x-1)^5}.$$

So $y' = 0$ at $x = -1/2$ and $y'' = 0$ at $x = -1$. $x = -1/2$ is a minimum because y' is positive (y increasing) for $x < -1/2$ and negative (y decreasing) for $x > -1/2$. $x = -1$ is an inflection because y'' changes sign as x goes through -1.

---

6. [10pts] How many zeros does the function $f(x) = 2 - 3x + x^3$ have between 0 and 2?'
$f(0) = 1$, $f(2) = 4$. Inconclusive. $f'(x) = 3x^2 - 3$, vanishes at $x = \pm 1$ and nowhere else. $f'' = 3x$ is positive if $x > 0$ so $x = +1$ is a local minimum. Now $f(1) = 0$, and that is a minimum in [0,2], so there is only one root in that interval and that root is $x = 1$.

---

7. [10pts] Sketch the curve $y = \cos^2 2x$ for $0 \leq x \leq \pi$. Find the area bounded by the curve, the x-axis and the vertical lines $x = 0$ and $x = \pi$.
The function is periodic of period $\pi/2$ and is always positive.

$$\int_0^\pi \cos^2 2x dx = \int_0^\pi \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) = \pi/2.$$

---

8. [10 pts] Solve $\dfrac{dy}{dx} = \dfrac{-1}{x^2 y}$ with $y(1) = C > 0$. Sketch $y(x)$ for $x \geq 0$. For what values of C is $\lim_{x \to +\infty} y(x) \geq 0$?
$ydy = -dx/x^2 \longrightarrow y^2/2 = 1/x + D$, where D is an arbitrary constant. We want $y(1) = C$, i.e. $C^2/2 = 1 + D$ and $D = C^2/2 - 1$. So the solution is

$$y(x) = \sqrt{C^2 - 2 + 2/x}.$$

This solution requires $C^2 - 2 + 2/x \geq 0$ and $x \to +\infty$ is possible only if $C \geq \sqrt{2}$. This is the escape velocity problem rephrased in terms of x and y, y is velocity and x distance from the earth (after normalization). [4.3# 12]

EXTRA EXERCISE: for each of these questions, identify the closest problem in the list of suggested problems and the warmup exam. Locate the solutions of those closest problems in your personal (lecture and discussion) notes.