1. *[20 pts] Evaluate the following limits, showing the main steps in your reasoning. Any correct method will do.*

(1) $\lim_{h\to 0} \dfrac{(1+h)^{1/3}-1}{h} = 1/3$

(i) This is the derivative of $x^{1/3}$ at $x=1$, so the limit is 1/3.
(ii) Use algebra: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ with $a=(1+h)^{1/3}$, $b=1$, then multiply top and bottom by a^2+ab+b^2, etc...
(iii) Use geometric series: $1+q+q^2 = (q^3-1)/(q-1)$. The $(\cdot)^3$ is just what we need. We flip things around to get $q-1 = (q^3-1)/(1+q+q^2)$. Now with $q \equiv (1+h)^{1/3}$, so $q \to 1$ as $h \to 0$, etc...

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(2) $\lim_{x\to+\infty} \dfrac{\sqrt{3+4x^2}-\sqrt{3}}{x-\sqrt{3}} = 2$

We've seen this type of thing on the warmup: the x^2 overwhelms everything else on top and the growing x wins out over that stuck, tiny $\sqrt{3}$ on the bottom. More cleanly:

$$\lim_{x\to+\infty} \frac{\sqrt{3+4x^2}-\sqrt{3}}{x-\sqrt{3}} = \lim_{x\to+\infty} \frac{x\sqrt{3/x^2+4}-\sqrt{3}}{x(1-\sqrt{3}/x)} = \lim_{x\to+\infty} \frac{\sqrt{3/x^2+4}-\sqrt{3}/x}{(1-\sqrt{3}/x)} = 2/1.$$

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(3) $\lim_{t\to 0} \left(t^2 \sin\dfrac{1}{t^3}\right) = 0$

We've talked about this a lot in class and on the warmup. It's that squeezed-sandwich thing again. The sin is $-1 \le \sin(\cdot) \le 1$ as always, so $-t^2 \le t^2\sin(\cdot) \le t^2$ and the function is squeezed $\to 0$ as $t \to 0$.

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(4) $\lim_{x\to 0} \dfrac{\sin x}{x+1-\cos x} = 1.$

It's that figure we've drawn many times again! If you draw the unit circle then the angle x, the $\sin x$ and $1-\cos x$, we expect that $1-\cos x \to 0$ much faster than x and $\sin x$. There's many ways to show this. The simplest is to first flip the function:

$$\lim_{x\to 0} \frac{x+1-\cos x}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} + \lim_{x\to 0} \frac{1-\cos x}{\sin x}$$

and we have studied both of those in class. See your notes.

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(5) $\lim_{\theta\to\pi/2} \frac{\cos\theta}{\pi-2\theta} = 1/2.$

If you visualize sin and cos, they're the same except for a $\pi/2$ shift. Indeed, let $x = \pi/2-\theta$ then

$$\lim_{\theta\to\pi/2} \frac{\cos\theta}{\pi-2\theta} = \lim_{x\to 0} \frac{\cos(\pi/2-x)}{2x} = \lim_{x\to 0} \frac{\sin x}{2x} = 1/2.$$

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2. *[20pts] Evaluate dy/dx for each of the following. Your answer should be in term of x.*

(1) $y^2 = \cos^2 4x + \sin^2 4x \equiv 1$, so $y' = 0$.

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(2) $x = 3+4t^3, \quad y = 5+\cos t,$

$$dx/dt = 12t^2, \qquad dy/dt = -\sin t$$
$$dy/dx = (dy/dt)(dt/dx) = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{12t^2}.$$

and, solving $x = 3 + 4t^3$ for t: $t = ((x-3)/4)^{1/3}$ so

$$dy/dx = \frac{-\sin\left(\frac{x-3}{4}\right)^{1/3}}{12\left(\frac{x-3}{4}\right)^{2/3}}.$$

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(3) $y = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$ so

$$dy/dx = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

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(4) $y = z^3 + 3z + 7, \quad z = \sqrt{x^2+1}.$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = (3z^2+3)\frac{x}{\sqrt{x^2+1}} = \frac{3x(x^2+2)}{\sqrt{x^2+1}}.$$

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3. *[10pts] Determine the constant C such that the curve $y = C - x^2$ is tangent to the curve $y = x$. Sketch the curves.*
LOOK at SUGGESTED PROBLEM 2.3 # 27!
$y = C - x^2$ is an upside-down parabola that goes through its maximum $y = C$ at $x = 0$. Its slope is $y' = -2x$. The slope of $y = x$ is $y' = 1$. So they have the same slope only at $x = -1/2$.
Now, to be tangent they also must go through the same point. That point is $y = x = -1/2$ and $y = C - 1/4$. So we need $C - 1/4 = -1/2$, or $C = -1/4$.

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4. *[10pts] Find the line(s) tangent to the curve $y = x^5 - x$ where the slope is equal to 4. What is the smallest value that this slope can ever have and where on the curve does the slope equal this smallest value? Sketch the curve.*
LOOK at SUGGESTED PROBLEM 2.3 # 18!
x^5 wins for large $|x|$, so y shoots off to $+\infty$ as $x \to +\infty$ and to $-\infty$ as $x \to -\infty$. But near $x = 0$, the $-x$ wins over the tiny x^5. So the curve goes through the origin $x = y = 0$, its slope is -1 there then it curves up or down depending on whether x is positive or negative.
$y' = 5x^4 - 1$ is equal to 4 if $5x^4 - 1 = 4$, or $x = \pm 1$, and $y = 0$ at both of those points. So the tangent lines are $y = 4(x-1)$ and $y = 4(x+1)$.
The minimum slope $y' = 5x^4 - 1$ is -1 at $x = 0$.

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5. *[15 pts] You want to give an apple to your TA before she grades this exam. She's up in Van Vleck at a height of H and you're at ground level. You throw the apple straight up in the air with initial velocity v_0, so the apple travels upwards according to Newton's laws:*

$$h(t) = v_0\, t - \frac{g}{2}t^2$$

where t is time, $h(t)$ is the height of the apple above the ground at time t and g is the acceleration of gravity ($g \approx 10$ m/sec^2).
(a) At what times is the apple at your TA's height H?
(b) What is the velocity of the apple when it goes by your TA at height H?
(c) What is the optimum velocity v_0 at which you should throw the apple?
(d) She can only reach the apple within the height interval $[H-l, H+l]$. What approximate amount of time does she have to catch the apple if $v_0 = 40m/sec$, $H = 35m$ and $l = 0.6m$?

LOOK AT SECTION 1.9 and all suggested sect 1.9 problems. See also SUGGESTED PROBLEM 2.2 # 3 AND 2.2 #16!

(a) $H = v_0 t - gt^2/2$ so $t_H = \left(v_0 \pm \sqrt{v_0^2 - 2gH}\right)/g$.

(b) $v(t) = dh/dt = v_0 - gt$ so $v(t_H) = \pm\sqrt{v_0^2 - 2gH}$.

(c) The optimum is when the apple stops at H, for an instant, so $v(t_H) = 0$, or $v_0 = \sqrt{2gH}$.

(d)

$$\frac{dh}{dt}(t_H) = \pm\sqrt{v_0^2 - 2gH} \Rightarrow \frac{dt}{dh}(H) = \frac{1}{\pm\sqrt{v_0^2 - 2gH}} \approx \frac{\Delta t}{\Delta h},$$

so your TA has about

$$\Delta t \approx \frac{\Delta h}{\sqrt{v_0^2 - 2gH}}$$

time to catch it on the way up and the same amount on the way down.

Now $\Delta h = 2l = 1.2$, $H = 35$, $v_0 = 40$, $g = 10$ so $\Delta t = 1.2/\sqrt{1600 - 700} = 1.2/\sqrt{900} = 1.2/30 = 0.04$. Not much time: 4 hundredths of a second!

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6. *[15pts] The curve defined by the equation $x^3 + y^3 = 6xy$ obviously goes through the origin and is symmetric about the $y = x$ line.*

(a) Find the other point(s) at which it intersects the $y = x$ line.

(b) Find the slope of the tangent to the curve at that (those) point(s).

(c) Find a parametric representation for the curve that explicitly defines x and y.

(d) Does the curve go to infinity? Explain.

(a) $y = x \Rightarrow 2x^3 = 6x^2$ so $x = y = 3$.

(b) $3x^2 + 3y^2 y' = 6y + 6xy'$, or $y' = (6y - 3x^2)/(3y^2 - 6y)$ and at $x = y = 3$, we get $y' = (18 - 27)/(27 - 18) = -1$.

(c) Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$ then the equation becomes $r^3(\cos^3\theta + \sin^3\theta) = 6r^2\sin\theta\cos\theta$, or

$$r = \frac{6\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}.$$

Going back to x and y:

$$\begin{cases} x = r\cos\theta = \dfrac{6\sin\theta\cos^2\theta}{\cos^3\theta + \sin^3\theta}, \\ y = r\sin\theta = \dfrac{6\sin^2\theta\cos\theta}{\cos^3\theta + \sin^3\theta}. \end{cases}$$

This is again very symmetric and can be further simplified if you know your trig. Divide top and bottom by $\cos^3\theta$, use the identity $1/\cos^2\theta = 1 + \tan^2\theta$ then let $t = \tan\theta$ and you get the friendlier form:

$$\begin{cases} x = \dfrac{6t}{1 + t^3}, \\ y = \dfrac{6t^2}{1 + t^3}. \end{cases}$$

The parameter t can have any value except -1.

This is a famous curve called the *folium of Descartes*. See the web site

`http://www-groups.dcs.st-and.ac.uk/~history/Java/index.html`

(d) It's pretty clear using the parametric form that x and $y \to \infty$ as $t \to -1$. With θ as the parameter x and $y \to \infty$ when $\cos\theta \to -\sin\theta$, i.e. when $\theta \to -\pi/4 + n\pi$. We can also deduce this from the algebraic equation $x^3 + y^3 = 6xy$. If $x \to \infty$ then there's a good chance that two of the three terms in the equation will run away together, leaving the third one in the dust. This could be $y^3 \approx -x^3$ with $|6xy| \ll |x^3|$ or the other possibility is that $x^3 \approx 6xy$ with $|y^3| \ll |x^3|$. This second case would imply $y \approx x^2/6$, but then $|y^3| \approx |x^6|/6^3$ which is certainly not much less than $|x^3|$ as $x \to \infty$! This assumption is not self-consistent. The first case implies $y \approx -x$ and indeed $|6xy| \approx |6x^2| \ll |x^3|$ as $|x| \to \infty$. So the curve does shoot to infinity and gets closer and closer to the line $y = -x$.

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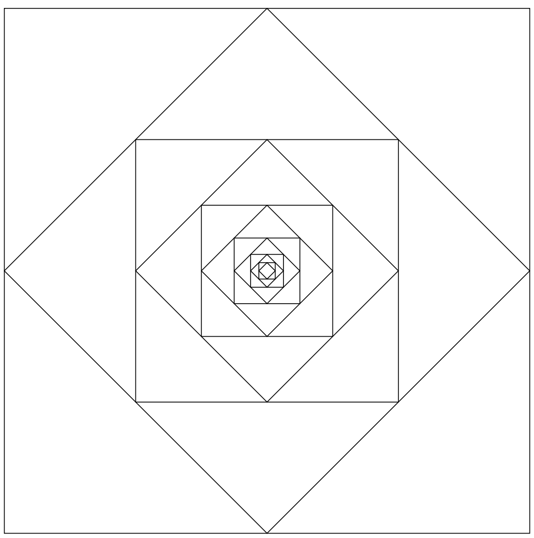
7. [10pts] *Starting with a square with sides of length L join the middle of each edge to create a new square, then join the middle of the edges of that new square to create another and so on indefinitely. What is the limit of the sum of the areas of all the squares?*



The first square has area $A_1 = L^2$, the second one has area $A_2 = A_1/2$, the third one has area $A_3 = A_2/2 = A_1/2^2$, etc... The area of any one square is one half the area of its predecessor. So the sum of the areas is

$$L^2\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots\right) =$$

$$\lim_{n\to\infty} L^2\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right) =$$

$$\lim_{n\to\infty} L^2 \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = 2L^2,$$

where we have used the summation formula for a geometric series:

$$1 + q + q^2 + q^3 + \cdots + q^n = \frac{q^{n+1} - 1}{q - 1}.$$