[2pts] Why should you have called your mother today?: MOTHER's DAY!

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1. [40 pts] Evaluate the following expressions.

(a) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt{4x^2-7}}{x} = \lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x^2}}-\sqrt{4-\frac{7}{x^2}}\right) = 1-2=-1$

(b) $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+}\frac{x^{-1}}{(-1/2)x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = 0^-$

(c) $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2} = \lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x} = \lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2} = \frac{n^2-m^2}{2}$

(d) $\displaystyle\frac{d}{dx}\int_1^8\frac{\cos(\ln x)}{x^3}\,dx = \frac{d}{dx}constant = 0$

(e) $\displaystyle\int_0^4\frac{t}{\sqrt{2t+1}}\,dt =$ (**done in class, in two ways, May 4th**)

Let $u=\sqrt{2t+1}$, then $du = dt/\sqrt{2t+1}$ and $t=(u^2-1)/2$. Then $\displaystyle\int_0^4\frac{t}{\sqrt{2t+1}}\,dt = \int_1^3\frac{u^2-1}{2}\,du = \left(\frac{u^3}{6}-\frac{u}{2}\right)\Big]_1^3 = 10/3.$

(f) $\displaystyle\int_0^{|b|}x\sqrt{x^2+b^2}\,dx =$ **Due may 3rd (#66 in 5.5).**

Let $u=x^2+b^2$, then $du=2xdx$ and

$\displaystyle\int_0^{|b|}x\sqrt{x^2+b^2}\,dx = \int_{b^2}^{2b^2}\frac{\sqrt{u}}{2}du = \frac{u^{3/2}}{3}\Big]_{b^2}^{2b^2} = |b|^3\left(\frac{\sqrt{8}-1}{3}\right).$

(g) $\displaystyle\int_0^{\pi/2}\frac{d}{dx}\left(\sin x\,\cos\frac{x}{2}\right)dx = \left(\sin x\,\cos\frac{x}{2}\right)\Big]_0^{\pi/2} = \cos\pi/4 = 1/\sqrt{2}.$

(**direct application of fundamental theorem of calculus**).

(h) $\displaystyle\int_0^{2\pi}|\sin x|\,dx = 2\int_0^{\pi}\sin x dx = 4.$

(i) Derivative of $\displaystyle\int_0^v\cos x^3dx = \cos v^3.$ **Due May 3rd, #8 in 5.4.**

(**direct application of fundamental theorem of calculus**).

(j) Derivative of $\displaystyle\int_{3x}^{\pi x^2}\sin(u^2)du = 2\pi x\sin(\pi^2x^4)-3\sin(9x^2).$ **Due may 3rd, #88 in 5.4.**

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2. [15pts] Approximate the natural log, $\ln x$, by a 2nd order polynomial that has the same value, the same derivative and the same 2nd derivative as $\ln x$ at $x=1$. Use your approximation to estimate ln(1.1) (yes, *without* your calculator!). Finally, draw a sketch of both $\ln x$ and your 2nd order polynomial approximation.

$(\ln x)' = 1/x$, $(\ln x)'' = -1/x^2$, thus at $x = 1$: $\ln x = 0$, $(\ln x)' = 1$ and $(\ln x)'' = -1$.
Want to find $P_2(x) = Ax^2 + Bx + C$ which has the same value, derivative and 2nd derivative as $\ln x$ at $x = 1$. This determines A, B, C.
In general

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \quad (Sect.\, 2.9)$$

So, $\ln x \approx (x-1) - \frac{1}{2}(x-1)^2$ near $x = 1$ and $\ln 1.1 \approx 0.1 - 0.01/2 = 0.1 - 0.005 = 0.095$.

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3. [10pts] The number of (real) roots of the equation $x^4 + 4x + a = 0$ is a function of a, call it $N(a)$. Determine $N(a)$ completely.

 # 22 in 4.2 DUE Apr 7. Also, similar (but easier!) to #3 on EXAM 2.

What kind of a curve is $x^4 + 4x + a = 0$? *Try to sketch it as a function of x!* It goes to $+\infty$ as $x \to \pm\infty$. It goes through a at $x = 0$. So it must have a minimum. Where? $\to 4x^3 + 4 = 0 \Rightarrow x = -1$. That's the only minimum and it is indeed a minimum becuase $f' < 0$ for $x < -1$ and $f' > 0$ for $x > -1$. So it's a global minimum. Its value is $f(-1) = 1 - 4 + a = a - 3$.

If the minimum is LARGER than 0 then there are no roots obviously. If the min is LESS than zero there are two roots.
So number of roots as a function of a, $N(a)$, is $N(a) = 2$ if $a < 3$, $N(a) = 1$ if $a = 3$ and $N(a) = 0$ if $a > 3$.

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4. [20pts] (a) State **two** ways to check whether a differentiable function $f(x)$ has a minimum at a point a. Find the minimum value(s) of $\sqrt{x}\ln x$, if they exist.

(1) $f'(a) = 0$ and f' goes from $f' < 0$ to $f' > 0$ as x crosses a.
(2) $f'(a) = 0$ and $f''(a) > 0$.
(3) $(\sqrt{x}\ln x)' = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x} = \dfrac{1}{2\sqrt{x}}(\ln x + 2) \longrightarrow = 0$ when $\ln x = -2$, i.e. $x = e^{-2}$, only minimum (f' goes from < 0 to > 0 as x crosses e^{-2}). Thus the minimum of $\sqrt{x}\ln x$ is $-2/e$.

(b) What happens at an *inflection point*? Find all the inflection point(s) of $x^4 e^x$, if they exist.

(1) 2nd derivative changes sign. Curve goes from concave up to concave down, or vice-versa.
(2) $(x^4e^x)'' = \cdots = (x^4 + 8x^3 + 12x^2)e^x = x^2(x^2 + 8x + 12)e^x$ vanishes at $x = 0$, $x = -2$ and $x = -6$. However $x = 0$ is NOT an inflection point, the concavity does not change, the sign of f'' does not change as x goes through zero. The other 2 points are inflections.

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5. [10pts] Sketch the graph of a function that satisfies all the following conditions

- $f(0) = 0$, $f(-1) = 1$, $f'(-1) = 0$, $\lim_{x \to +\infty} f(x) = 1$,
- $f''(x) > 0$ on $(-\infty, -1)$,

- $f''(x) < 0$ on $(-1, 0)$ and $(0, \infty)$
- $f'(x) > 0$ for $x > 0$.

This is basically # 30 in 4.4. **Due Apr 7**.

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6. [15pts] A wire of length L is cut into two pieces. One piece is bent into a square and the other into a circle. The perimeter of the square cannot be smaller than $L/2$. How should the wire be cut to (a) maximize and (b) minimize the total enclosed area? (it will be helpful to know that $3 < \pi < 4$).

Done in class. Very similar to # 7 on Exam 2.

Let x be length of piece that's bent into square (of side $x/4$ then), so $L - x$ is bent into circle (of radius $(L-x)/(2\pi)$ then).
$\Rightarrow$ Area of square $= x^2/16$, Area of circle $= (L-x)^2/(4\pi)$.
Total area:

$$A = \frac{x^2}{16} + \frac{(L-x)^2}{4\pi}$$

$$\Rightarrow \frac{dA}{dx} = \frac{x}{8} - \frac{(L-x)}{2\pi} = x\left(\frac{1}{8} + \frac{1}{2\pi}\right) - \frac{L}{2\pi}$$

Thus $A' = 0$ when $x = \dfrac{L}{\left(1 + \frac{2\pi}{8}\right)} > \dfrac{L}{2}$.
$A(x)$ is a parabola, concave upward, so $A' = 0$ corresponds to a *minimum*. (or: A' goes from $A' < 0$ to $A' > 0$).
The maximum comes from the fact that x is limited in range, *Sketch $A(x)$!!*. Here $L/2 \leq x \leq L$. So the maximum is either $A(L/2) = L^2/64 + L^2/(16\pi)$ or $A(L) = L^2/16$. Which is largest? well, $A(L/2) = L^2/16 \ \ (1/4 + 1/\pi)$ is less than $A(L) = L^2/16$ because

$$\frac{1}{4} + \frac{1}{\pi} < 1.$$

So the maximum occurs at $x = L$ and is equal to $L^2/16$.

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7. [10pts] Draw a nice sketch of $f(x) = \sqrt{R^2 - x^2}$, then calculate $\int_{-3R}^{4R} f(x)dx$.

This is a half-circle of radius R! $f(x) \geq 0$ so the integral is the *area under the curve* $= 1/2$ area of circle $= \pi R^2/2$.

A couple of people said that the function was not defined beyond $|x| > R$ (true) and thus the integral was not defined. This was accepted as a correct answer as well.