Feedback on vertical velocity. Rotation, convection, self-sustaining process.

Fabian Waleffe

Notes by Andrew Crosby, Keiji Kimura, Adele Morrison
Revised by FW

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Abstract

We have shown in the previous lectures that the $-vU'$ term in the u equation, that is $\partial_t u = -vU' + \cdots$, is the key term leading to momentum transport $-\overline{uv} \sim \overline{vv}U't$ and perturbation energy production $-\overline{uv}\,U' \sim \overline{vv}(U')^2 t > 0$. This term is the redistribution of streamwise velocity that releases energy from the background shear to enable bifurcation to turbulent flow. However, in shear flows we have not yet identified a mechanism that can feedback from the u fluctuation to v, thus v creates just the right u through the $-vU'$ advection term, but how is v sustained? In this lecture, we first review two *linear* mechanisms of feedback on v involving extra physics, (1) through the Coriolis term in rotating shear flow, (2) through buoyancy in Rayleigh-Bénard convection. We derive the famous Lorenz model for convection, then consider a similar model for shear flows that illustrates the mechanisms involved in the *nonlinear* feedback from u to v, yielding a self-sustaining process for shear flows $v \to u \to \cdots \to v$. This is the model that was already discussed in lecture 1.

1 Redistribution of streamwise velocity

In this lecture we will consider the mechanisms that lead to a 3D nonlinear self-sustaining process in shear flows. Let us first consider what this may look like in plane Couette flow. If we introduce a perturbation in the y-direction (in this case $v > 0$), such as a cross-stream jet, the perturbation will create so-called *streamwise rolls,* as shown in Figure 1, that is a flow $\mathbf{v} = (0, v(y,z), w(y,z))$ as a result of incompressibility and the boundary conditions.[1] The streamwise rolls redistribute the streamwise velocity, by advecting negative momentum ($u < 0$) up and towards $z = 0$ (where $z = 0$ is defined as the position of the jet) and positive momentum ($u > 0$) down and away from $z = 0$.

The redistribution of the streamwise momentum results in a pattern of so-called *streaks* in u, with low streamwise velocity at the position of the jet. 'Streaks' refers to the patterns made by hydrogen bubbles released in the near wall region of turbulent shear flows. In theory, they refer to spanwise fluctuations of the streamwise velocity u, that is the departure of the x averaged u, $\overline{u}^x(y,z)$ from the x and z averaged u, $\overline{u}(y)$,

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[1]Streamwise rolls have their axis in the streamwise direction and they are streamwise-independent. Their horizontal wavevector $\mathbf{k}_H = (k_x, k_z) = (0, k_z)$ is actually pointing in the spanwise direction z.
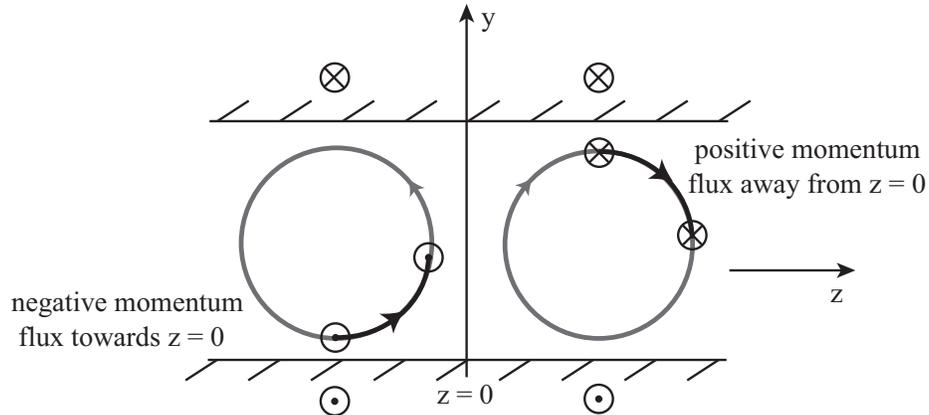
Figure 1: Streamwise rolls redistribute the momentum of the mean shear from the bottom wall up and toward $z = 0$ and from the top wall down and away from $z = 0$. The mean shear flow is into and out of the screen.

so the streaks are defined as $\bar{u}^x(y, z) - \bar{u}(y)$ (with t implicit). Figure 2 shows that the perturbation induces streaks of faster and slower streamwise flow. The profile of u now has inflection points, but in the spanwise z direction, and so we may expect it to be unstable.
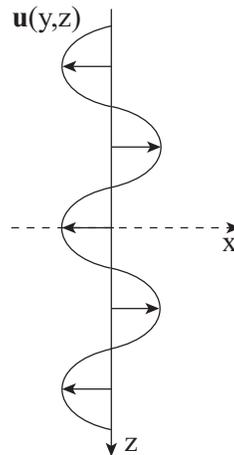

Figure 2: Top view: Streamwise rolls redistribute the mean shear creating streaks.

However, in order for the flow to bifurcate from the laminar flow, it is not enough to have rolls $(0, v(y, z), w(y, z))$ creating unstable streaky flow $(u(y, z), 0, 0)$, we need feedback from the streak instability into the rolls sufficient to lead to a self-sustaining process. To investigate whether a mechanism exists to feedback on the v perturbation, let us consider the energy stability discussed in earlier lectures. If the perturbations are independent of the streamwise x direction, we can separate out the streamwise from the

cross-stream components as follows:

$$\frac{d}{dt}\int \frac{u^2}{2}dV = \int (-uvU')dV - \frac{1}{R}\int |\boldsymbol{\nabla}u|^2 dV \tag{1}$$

$$\frac{d}{dt}\int \frac{v^2+w^2}{2}dV = -\frac{1}{R}\int \left(|\boldsymbol{\nabla}v|^2 + |\boldsymbol{\nabla}w|^2\right)dV \tag{2}$$

It is evident from eqn. (2) that there is dissipation, but no production, of the cross-stream components. Therefore a further mechanism is needed if the rolls and streaks created by the initial perturbation are to be sustained.

We can further investigate the feedbacks between the streamwise rolls and streaks by considering the streamwise and cross-stream linearized momentum equations eliminating any x dependence (see lectures 1 and 4)

$$\partial_t u - \frac{1}{R}\nabla^2 u = -vU' \tag{3}$$

$$\partial_t v - \frac{1}{R}\nabla^2 v + \partial_y p = 0 \tag{4}$$

$$\partial_t w - \frac{1}{R}\nabla^2 w + \partial_z p = 0 \tag{5}$$

with the pressure $p(y,z,t)$ enforcing $\partial_y v + \partial_z w = 0$ and where $U' = dU/dy$ is the shear rate of the laminar base flow. Equation 3 further emphasises the creation of streaks from the streamwise rolls; an updraft $v > 0$ creates $u < 0$, while a downdraft $v < 0$ creates $u > 0$, assuming $U' > 0$. However the cross-stream velocities v, w are decoupled from the streamwise velocity u and therefore the rolls will decay.

2   Rotation induced shear instability

One method of creating feedback from u onto v is to add rotation to our system of plane Couette flow. Let us consider rotation of the form $\Omega\hat{\mathbf{z}}$, with $\Omega > 0$. The linearized momentum equations in the rotating frame involve the *Coriolis force* $-2\Omega\hat{\mathbf{z}} \times \mathbf{v} = (2\Omega v, -2\Omega u, 0)$ (the centrifugal force is absorbed into the pressure gradient) and read

$$\partial_t u - \frac{1}{R}\nabla^2 u = -vU' + 2\Omega v \tag{6}$$

$$\partial_t v - \frac{1}{R}\nabla^2 v + \partial_y p = -2\Omega u \tag{7}$$

with the same w equation (5). The Coriolis force adds a feedback from u to v. The Coriolis force is energy conserving since it is always orthogonal to $\mathbf{v}$. However, in the presence of background shear U', the advection of U by v creates u that is rotated into v by the Coriolis term and this can lead to a linear instability. As shown previously in Figure 1, a positive perturbation in v will lead to a negative perturbation in u, assuming $U' > 2\Omega$ via the advection of mean shear (6). Coriolis $-2\Omega u$ in the v equation (7) turns this negative u into a positive feedback on v, thereby sustaining the rolls and destabilizing the flow. Ignoring diffusion and the pressure gradient, equations (6) and (7) suggest instability when

$$2\Omega(U' - 2\Omega) > 0, \tag{8}$$

for a base flow $U(y)\hat{\mathbf{x}}$ and rotation $\Omega\hat{\mathbf{z}}$. Thus, too much rotation $|U'| < 2|\Omega|$ will lead to stability and any rotation in the direction of the shear vorticity (*i.e.* $U'\Omega < 0$) will also stabilize, but a bit of rotation in the direction *opposite* to the shear flow vorticity

(*i.e.* $U' > 2\Omega > 0$ or $U' < 2\Omega < 0$) will lead to linear instability as a result of shear redistribution and the Coriolis force.

This simple outline of rotating plane Couette flow is a way of understanding the classic 'centrifugal' instability seen in Taylor-Couette flow, in which fluid contained within two concentric cylinders is unstable *when the inner cylinder rotates faster than the outer cylinder,* and indicate that that classic instability should perhaps be called 'Coriolis instability' instead of 'centrifugal'. The rolls and streaks in Taylor-Couette flow are shown in Figure 3. Placing a plane Couette flow on a rotating table leads to similar 'Taylor vortices' as done in experiments by Tillmark and Alfredsson.
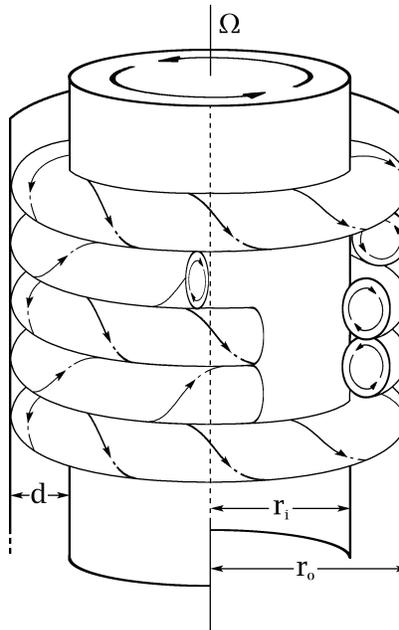

Figure 3: Toroidal vortices in Taylor-Couette flow (by M. Minbiole and R. M. Lueptow)

3 Thermal convection and the Lorenz model

A second method of creating feedback onto the cross-stream velocity perturbations v, w is to add a thermal gradient. We consider the Boussinesq thermal convection between planes, *i.e.* Rayleigh-Bénard convection (see Figure 4), especially in the weakly nonlinear regime, and derive qualitatively the reduced system of ODEs known as the Lorenz model.

The governing equations of Boussinesq thermal convection between planes are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = g\alpha T \hat{\mathbf{y}} + \nu \nabla^2 \mathbf{u}, \tag{9}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{10}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa \nabla^2 T, \tag{11}$$
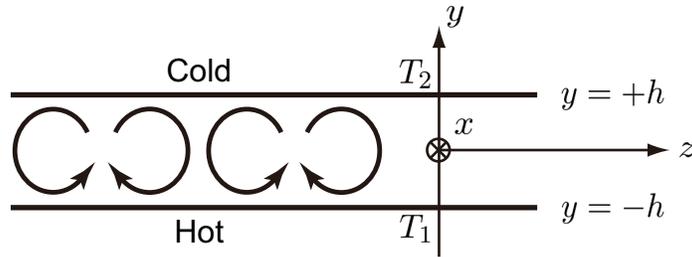
Figure 4: Schematic picture of Rayleigh-Bénard convection. Gravity acts downward in the $-\hat{\mathbf{y}}$ direction.

where $\mathbf{u}$ is the velocity, $-g\hat{\mathbf{y}}$ is the acceleration of gravity, T the temperature[2] departure from some mean temperature so that the density $\rho \approx \rho_0(1 - \alpha T)$ with $\alpha \geq 0$ the thermal expansion coefficient, ν the kinematic viscosity and κ the thermal diffusivity. The distance between the two plates is $H = 2h$ and the temperature of the lower plate is T_1 at $y = -h$ and the upper plate is at T_2 at $y = h$, with $T_1 + T_2 = 0$. Thermal convection occurs when the temperature difference $\Delta T = T_1 - T_2$ is larger than a certain threshold. The base temperature profile is

$$T = T_c = -\frac{\Delta T}{H}y, \tag{12}$$

which the conductive state solution of (9), (10) and (11) with $\mathbf{u} = 0$, for all values of the parameters. Buoyancy $g\alpha T_c\,\hat{\mathbf{y}}$ in (9) is balanced by an hydrostatic pressure.

(1) Insert rolls. We choose the boundary conditions as free-slip at the walls, that is $v = \partial_y w = \partial_y^2 v = 0$, this is a physically reasonable boundary condition that is mathematically convenient since it allows the representation of the velocity by Fourier modes. We insert a 'streamwise roll' into the flow that can be taken as

$$v = V(t)\cos{(\beta y)}\cos(\gamma z), \tag{13}$$

where $v = \hat{\mathbf{y}} \cdot \mathbf{u}$ and $\beta = \pi/(2h)$ so $v = \partial_y^2 v = 0$ at $y = \pm h$ and γ is an arbitrary wavenumber. We choose the flow in the (y, z) plane to match the shear flow problem (fig. 1). From incompressibility $\partial_y v + \partial_z w = 0$, the z velocity $w = \hat{\mathbf{z}} \cdot \mathbf{u}$ must be

$$w = \frac{\beta}{\gamma}V(t)\sin{(\beta y)}\sin(\gamma z), \tag{14}$$

with $\partial_y w = 0$ at $y = \pm h$ since $\beta = \pi/(2h)$. So this $\mathbf{u} = (0, v, w)$ flow (13), (14) as the shape of the rolls sketched in Figure 4 and satisfies the boundary conditions $v = \partial_y w = \partial_y^2 v = 0$ at $y = \pm h$.

(2) Rolls redistribute temperature. Let $T = T_c(y) + \tilde{T}(y, z, t)$, that is, the conductive profile (12) plus a perturbation $\tilde{T}$. Then, (11) becomes

$$\frac{\partial\tilde{T}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\tilde{T} = -v\frac{dT_c}{dy} + \kappa\nabla^2 T \tag{15}$$

which is entirely similar to the streamwise velocity perturbation equation in shear flows (3) (and earlier lectures). Inserting the v mode (13) into the temperature equation (15)

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[2]Here, T is *temperature*, not the perturbation Reynolds stress that appeared in earlier lectures!

with $T_c = -y\Delta T/H$ shows that the $-vdT_c/dy = v\,\Delta T/H$ term generates a thermal perturbation of the form

$$\tilde{T} = T_{11}(t)\cos{(\beta y)}\cos(\gamma z) + \cdots, \tag{16}$$

that is the rolls (13) redistribute the linear temperature profile (12) inducing a temperature fluctuation $\tilde{T}$ that has the same spatial form as v. The nonlinear term $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ for (13), (14) can be absorbed into the pressure gradient,[3] so it does not distort the velocity field. If it did (for different rolls, say for no-slip), we would assume at this stage that the rolls are weak so $\boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u})$ is small.

(3) *Temperature fluctuation feedback onto rolls.* The temperature fluctuation $\tilde{T}$ (16) yields a buoyancy fluctuation $g\alpha\tilde{T}\hat{\mathbf{y}}$ in the momentum equation (9) that perfectly feeds back on the v mode (13). These $V \to T_{11} \to V$ interactions through the base state $T_c(y)$ will lead to thermal instability provided this feedback can overcome the viscous and thermal damping of the V and T_{11} modes, as shown below (27).

(4) *Mean temperature gradient reduction.* The 'next' effect[4] is the interaction between the rolls (13) and the induced fluctuation $\tilde{T}$ (16), in the temperature equation (15). This interaction arises from the advection term $-\mathbf{u}\cdot\boldsymbol{\nabla}\tilde{T}$ in (15) for v, w and $\tilde{T}$ as in (13), (14), (16), then after a simple calculation

$$-\mathbf{u}\cdot\boldsymbol{\nabla}\tilde{T} = -v\frac{\partial\tilde{T}}{\partial y} - w\frac{\partial\tilde{T}}{\partial z} = \frac{\beta}{2}\,V\,T_{11}\,\sin(2\beta y) + \cdots \tag{17}$$

so this interaction generates a $\sin(2\beta y)$ temperature fluctuation, which is a modification of the mean temperature profile $\overline{T}(y,t)$, that is the z (and x) averaged temperature profile. We label this mode the T_{20} mode since it is a temperature mode with y wavenumber 2β and z wavenumber 0, hence the *'two-zero'* (20) mode. This reduction of the mean temperature gradient will lead to a reduction of the T_{11} forcing and saturation of the instability.

Therefore, the temperature distribution has the form

$$T = -y\frac{\Delta T}{H} + T_{11}(t)\cos{(\beta y)}\cos(\gamma z) + T_{20}(t)\sin(2\beta y) + \cdots. \tag{18}$$

together with the velocity field $\mathbf{u} = (0, v, w)$ with v and w as in (13), (14). The cause-effect chain that we have in mind being $V \longrightarrow T_{11} \longrightarrow V \Longrightarrow T_{20} \Longrightarrow T_{11}$, where as in the scenarios of lecture 1, a $\longrightarrow$ indicates a linear interaction (that is between the base flow and a fluctuation) and a $\Longrightarrow$ a nonlinear interaction between fluctuations. Note that amplitude-wise $T_{11} \sim V$, $T_{20} \sim VT_{11} \sim V^2$ and the feedback from V and T_{20} onto T_{11} is of order $VT_{20} \sim V^3$.

We have identified a physically consistent set of modal interactions and can truncate the expansion at this level. In general, this truncation will be valid only for sufficiently small amplitude V since other modes are generated. We truncate the above formulation using what is known as the Galerkin truncation or projection, that is we substitute these expansions for v, w and $\tilde{T}$ into the equations and throw away higher order modes such as $\cos 2\gamma z$, $\cos 3\beta y$, etc. In the Galerkin truncated system, the dependent variables are

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[3] $\boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}) = \mathbf{u}\cdot\boldsymbol{\nabla}\omega\hat{\mathbf{x}}$ where $\omega = \partial_y w - \partial_z v$ for this 2D flow. Using a streamfunction $v = \partial_z\psi$, $w = -\partial_y\psi$ gives $\mathbf{u}\cdot\boldsymbol{\nabla}\omega = (\partial_y\psi\partial_z - \partial_z\psi\partial_y)(\partial_y^2 + \partial_z^2)\psi$. Now for (13), $\psi = (V/\gamma)\cos\beta y\sin\gamma z$ and $\nabla^2\psi = -(\beta^2 + \gamma^2)\psi$ so $\mathbf{u}\cdot\boldsymbol{\nabla}\omega = 0$. Thus $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\nabla}\chi$, indeed $\chi = \left(\cos 2\beta y - (\beta^2/\gamma^2)\cos 2\gamma z\right)V^2/4$.

[4] All these effects take place simultaneously but here we identify a cause and effect sequence, starting with a small stirring of the fluid, then analyzing the dynamical consequences of that stirring.

the modal amplitudes $V(t)$, $T_{11}(t)$ and $T_{20}(t)$, thus we reduce the complete set of PDEs (9), (10), (11) to 3 ODEs.

The temperature equation (15) is easy enough, but the **u** equation is complicated by the pressure gradient needed to enforce incompressibility. For 2D flow, we can eliminate pressure and **u** using a streamfunction, or as in the derivation of the Orr-Sommerfeld equation by taking $\hat{\mathbf{y}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla}(*)) = (\partial_y \boldsymbol{\nabla} - \hat{\mathbf{y}}\nabla^2) \cdot (*) \equiv \boldsymbol{P}_v \cdot (*)$, so $\boldsymbol{P}_v \cdot \mathbf{u} = -\nabla^2 v$ and $\boldsymbol{P}_v \cdot \boldsymbol{\nabla}\varphi = 0$. Applying this $\boldsymbol{P}_v$ operator to (9) yields

$$\partial_t \nabla^2 v = \nu \nabla^2 \nabla^2 v + g\alpha \nabla_\perp^2 T + \boldsymbol{P}_v \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}) \tag{19}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ and $\nabla_\perp^2 = \partial_x^2 + \partial_z^2$ and **u** follows from v and incompressibility in 2D flow, and $\boldsymbol{P}_v \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}) = 0$ for (13). Additional $\eta = \hat{\mathbf{y}} \cdot \boldsymbol{\nabla} \times \mathbf{u}$ and mean flow equations would be needed for general 3D flow but (19) together with (15) suffice for 2D flow. Then for v and T as in (13) and (18), equations (15) and (19) yield the governing equations for the modal amplitudes $V(t)$, $T_{11}(t)$ and $T_{20}(t)$,

$$\begin{aligned}
\frac{dV}{dt} + \nu_{11} V &= g_{11} T_{11}, \\
\frac{dT_{11}}{dt} + \kappa_{11} T_{11} &= V\frac{\Delta T}{H} - \beta V T_{20}, \\
\frac{dT_{20}}{dt} + \kappa_{20} T_{20} &= \frac{\beta}{2} V T_{11},
\end{aligned} \tag{20}$$

where

$$\nu_{11} = \nu(\beta^2 + \gamma^2), \quad \kappa_{11} = \kappa(\beta^2 + \gamma^2), \quad \kappa_{20} = \kappa(4\beta^2), \quad g_{11} = g\alpha\frac{\gamma^2}{\beta^2 + \gamma^2} \tag{21}$$

and $\beta = \pi/(2h) = \pi/H$. This is the famous Lorenz model of convection.[5] One solution of these equations is $V = T_{11} = T_{20} = 0$ which corresponds to the conductive state. Linearizing about that state yields

$$\begin{aligned}
\frac{dV}{dt} + \nu_{11} V &= g_{11} T_{11}, \\
\frac{dT_{11}}{dt} + \kappa_{11} T_{11} &= V\frac{\Delta T}{H},
\end{aligned} \tag{25}$$

which has an unstable mode $V(t) = e^{\lambda t}\hat{V}$, $T_{11}(t) = e^{\lambda t}\hat{T}_{11}$ if $g_{11}\Delta T/H > \nu_{11}\kappa_{11}$, that is for

$$\frac{g\alpha\Delta T}{H}\frac{\gamma^2}{\beta^2 + \gamma^2} > \nu\kappa(\beta^2 + \gamma^2)^2 \tag{26}$$

or in non-dimensional form with $\hat{\beta} = \beta H = \pi$ and $\hat{\gamma} = \gamma H$

$$Ra \equiv \frac{g\alpha\Delta T H^3}{\nu\kappa} > \frac{(\pi^2 + \hat{\gamma}^2)^3}{\hat{\gamma}^2} \geq \frac{27\pi^4}{4} \approx 657.5, \tag{27}$$

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[5]The standard non-dimensional form of the Lorenz model is

$$\frac{dx}{d\tau} = \sigma(y - x), \tag{22}$$

$$\frac{dy}{d\tau} = -y + rx - xz, \tag{23}$$

$$\frac{dz}{d\tau} = -bz + xy, \tag{24}$$

where $\tau = \kappa_{11}t$, $\sigma = \nu/\kappa$, $r = g_{11}(\Delta T)/(\nu_{11}\kappa_{11}H)$, $b = \kappa_{20}/\kappa_{11}$ and $x = \beta V/(\sqrt{2}\kappa_{11})$, $y = \beta/(\sqrt{2}\kappa_{11})(g_{11}/\nu_{11})T_{11}$, $z = (\beta/\kappa_{11})(g_{11}/\nu_{11})T_{20}$.
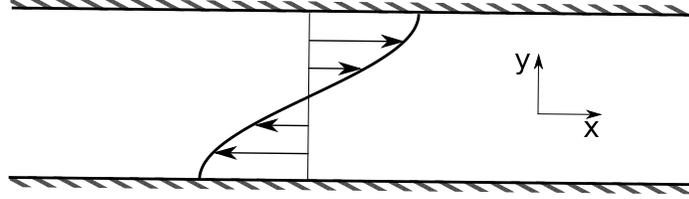
Figure 5: Wall-bounded Kolmogorov flow, $U = \sin\beta y$ with $\beta y = \pm\pi/2$ at the walls, is stable for all Reynolds numbers for free-slip perturbations, as proved in lecture 4.

where Ra is the non-dimensional *Rayleigh number* and the minimum $Ra = 27\pi^4/4 = Ra_c$ value for onset of convection with free-slip occurs at $\hat\gamma = \gamma H = \pi/\sqrt{2}$.

This, of course, is just the beginning of the story since the simple nonlinear model (20) has some rather interesting chaotic nonlinear dynamics as first studied by E. Lorenz (1963), and for the the full PDEs (9), (10), (11), this is just the onset of convection leading to multiscale turbulence for larger Ra, much beyond the range of validity of the simple Lorenz model.

4 SSP Model

In the previous sections we have illustrated instabilities that result from the interactions $v \to u \to v$ thanks to base velocity redistribution $-v\,dU/dy$ and the Coriolis force, and from the interaction $v \to T_{11} \to v$ thanks to mean temperature redistribution $-v\,dT/dy$ and buoyancy in convection. In bare shear flows, there is no direct linear feedback from $u \to v$, but there are more involved 3D nonlinear mechanisms that provide that crucial feedback. The entire set of mechanisms is called the *self-sustaining process* and we illustrate it here with a low order model similar in spirit to the Lorenz model of convection.

From the earlier energy stability result (Equation 2) we know that unless we have variation of the velocity in the x direction any perturbations must eventually die out (possibly after some initial transient growth) thus any successful model must include such variation.

The SSP model was already described in lecture 1 and was first described in [2, 3]. The model can be derived by Galerkin truncation from the full Navier-Stokes equations [4], and some mode linking.

Brief derivation

We consider a wall-bounded Kolmogorov flow with free-slip at the walls, figure 5. This will allow us to use Fourier modes to represent the flow and lead to a simpler and cleaner model than other shear flows with no-slip.

$$\mathbf{U}(y,t) = M(t)\sin(\beta y)\hat{\mathbf{x}} \tag{28}$$

with $\beta = \pi/2$ and the walls at $y = \pm 1$. We assume that this flow is maintained by a body force, so that the evolution of M in the absence of any perturbations is given by

$$\dot{M} + \frac{1}{R}M = \frac{1}{R}, \tag{29}$$

for which one solution is simply $M = 1$ and R is the Reynolds number. We proved in lecture 4 that this wall-bounded Kolmogorov flow is stable for all Reynolds numbers for free-slip perturbations.

As before we consider an initial perturbation consisting of rolls whose axes are in the x direction, see Figure 1, and flow in the y and z directions as follows:

$$v = V(t) \ \cos(\beta y) \cos(\gamma z),$$
$$w = \frac{\beta}{\gamma} V(t) \ \sin(\beta y) \sin(\gamma z) \tag{30}$$

this $V(t)$ mode satisfies incompressibility $\partial_y v + \partial_z w = 0$ and the free-slip boundary conditions $v = \partial_y w = \partial_y^2 v = 0$ at $\beta y = \pm \pi/2$. This is the same rolls as in the Lorenz model (13). The vertical component, v, of these perturbations advects momentum associated with the background flow and leads to a perturbation of the velocity in the x direction, which to leading order has structure

$$u = U(t) \cos(\gamma z), \tag{31}$$

these are the *streaks* and the need for this mode arises from the redistribution of the base shear (28) by the rolls (30), so

$$v\partial_y \mathbf{U} + w\partial_z \mathbf{U} = \beta V M \cos\gamma z \ (\cos\beta y)^2 \hat{\mathbf{x}} = \frac{\beta}{2} V M \cos\gamma z \ (1 + \cos 2\beta y) \ \hat{\mathbf{x}}.$$

The $\cos 2\beta y$ was dropped in the derivation in [4] but it might be better to keep it and define the streaks as $U(t) \cos\gamma z \ (1 + \cos 2\beta y)$. Moehlis *et al.* [1] have considered small modifications of the 8 mode model in [4] that includes a few such adjustments of the mode definitions. In any case, there is a VM forcing of U and likewise the advection by the rolls of the streaks $u\hat{\mathbf{x}}$ with u as in (31) gives

$$v\partial_y u + w\partial_z u = -\beta \ VU \ \sin\beta y \ (\sin\gamma z)^2 = -\frac{\beta}{2} VU \ \sin\beta y \ (1 - \cos 2\gamma z)$$

and this gives a negative feedback on the base flow (28). Again we truncate the $\cos 2\gamma z$ contribution.

At this level, our Galerkin truncation for the redistribution of the base flow (28) by the streamwise rolls (30) has the form

$$\begin{aligned}
\dot{M} + \tfrac{1}{R}M &= \tfrac{1}{R} &- UV \\
\dot{U} + \tfrac{1}{R}U &= &VM \\
\dot{V} + \tfrac{1}{R}V &= 0 &
\end{aligned} \tag{32}$$

where the dot $\dot{(\ )} \equiv d/dt$ and the coefficients have been set to 1 to highlight the structure of the interactions as clearly as possible. However, as our motion is still independent of the x direction, there is nothing to regenerate the vertical motion V and all perturbations will eventually die out. Viewed from above the flow now has strips of faster and slower flowing fluid, the streaks U (31) sketched in Figure 2, which have been generated by the advection of the mean shear M by the rolls V. This streaky flow has inflection points in the spanwise z direction and hence we might expect it to be unstable to an inflectional instability as discussed in lecture 4. This instability provides a mechanism for introducing the variation in the x direction that we know is vital to sustain the perturbations. This inflectional instability of $U \cos\gamma z\hat{\mathbf{x}}$ will consist of the growth of
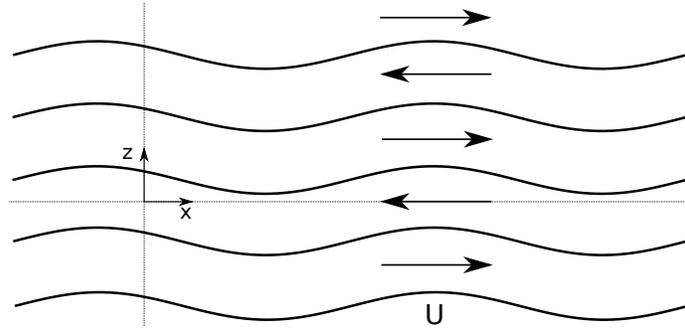
Figure 6: Inflectional instability viewed from below.

'streak-sloshing' mode, illustrated in Figure 6, that is most simply represented by a spanwise velocity perturbation periodic in x, that is

$$w = W(t)\cos(\alpha x) \tag{33}$$

(we could just as well choose $\sin \alpha x$, this is a phase choice that is inconsequential to the dynamics at this point). This spanwise perturbation should be added to the streamwise roll contribution (30) so the full spanwise velocity at this level of truncation is

$$w = W(t)\cos(\alpha x) + \frac{\beta}{\gamma}V(t)\ \sin(\beta y)\sin(\gamma z) \tag{34}$$

Mode $\boldsymbol{\Psi}_{100} = (0, 0, \cos \alpha x)$ in (33) can grow from the U streak mode $\boldsymbol{\Psi}_{001} = (\cos \gamma z, 0, 0)$ but only through interaction with the velocity mode $(-\gamma \cos \alpha x \sin \gamma z,\ 0,\ \alpha \sin \alpha x \cos \gamma z)$ labelled $\boldsymbol{\Psi}_{101}$ in [4]. The $\boldsymbol{\Psi}$ modes are solenoidal, $\boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = 0$, and satisfy the boundary conditions. The details are left to [4] but the conceptual result is that we now have an x dependent mode $W(t)$ and interactions between U and W such that W grows from an instability of U, so the 'forcing' of W is in the form of a UW. Conceptually, the dynamics reads

$$
\begin{array}{rclll}
\dot{M} + \frac{1}{R}M & = & \frac{1}{R} & -UV & \\
\dot{U} + \frac{1}{R}U & = & & VM & -W^2 \\
\dot{V} + \frac{1}{R}V & = & 0 & & \\
\dot{W} + \frac{1}{R}W & = & & & UW
\end{array} \tag{35}
$$

Unfortunately there is still no feedback on V so our perturbations must still die out. This is an important point rarely understood, *streak instability does not guarantee bifurcation from the laminar flow.* The streak instability could simply, as in model (35), destroy the streaks U created by V from M, and therefore accelerate the return of the flow to the laminar state $(M, U, V, W) = (1, 0, 0, 0)$.

In fact it is remarkably difficult to generate feedback on the streamwise rolls from the streak eigenmode interactions. This feedback requires the interaction of modes with opposite y-symmetry, *i.e.* an even mode interacting with an odd mode, and those modes arise from each other through interaction with the mean shear. Thus as observed in [4], the mean shear M plays the dual role of supplier of energy and momentum but also shaper of the streak instability in order to allow feedback on V from the nonlinear interaction of the streak instability eigenmode. It turns out that no less than five x-dependent modes are required before we can get feedback on V, [4, §III. A], giving an

8th order model. However we can simplify the model by kinematic linking of those 5 modes into a single complex mode of amplitude W, the details are spelled out in [4, §III. C] and the result conceptually is

$$
\begin{aligned}
\dot{M} + \tfrac{1}{R}M &= \tfrac{1}{R} & -UV & & & \\
\dot{U} + \tfrac{1}{R}U &= & VM & -W^2 & & \\
\dot{V} + \tfrac{1}{R}V &= 0 & & & W^2 & \\
\dot{W} + \tfrac{1}{R}W &= & & & UW & -VW
\end{aligned}
\tag{36}
$$

with, at last, feedback on V.

However, the derivation in [4] shows that these interactions are necessarily accompanied by a feedback from W to M, which can be interpreted as the unavoidable shearing of x dependent modes by the mean shear $M \sin \beta y \, \hat{\mathbf{x}}$ that tends to destroy the W mode and by conservation of energy, transfer that energy and momentum back to M. This is therefore a mean shear stabilizing term and it is key to the $1/R$ threshold discussed in lecture 1. The complete model [4, eqn. (20)] is then

$$
\begin{aligned}
\left(\frac{d}{dt} + \frac{\kappa_m^2}{R}\right) M &= \frac{\kappa_m^2}{R} & -\sigma_u\, UV & & & & +\sigma_m\, W^2 \\
\left(\frac{d}{dt} + \frac{\kappa_u^2}{R}\right) U &= & \sigma_u\, MV & -\sigma_w\, W^2 & & & \\
\left(\frac{d}{dt} + \frac{\kappa_v^2}{R}\right) V &= & & & \sigma_v\, W^2 & & \\
\left(\frac{d}{dt} + \frac{\kappa_w^2}{R}\right) W &= & & \sigma_w\, UW & -\sigma_v\, VW & -\sigma_m\, MW
\end{aligned}
\tag{37}
$$

where all κ^2 and σ are positive constants and $R > 0$ is the Reynolds number. This model (37) has a laminar flow $(M, U, V, W) = (1, 0, 0, 0)$, that is linearly stable for all Reynolds number. The nonlinear interactions conserve energy and

$$
\frac{d}{dt}\left(\frac{M^2 + U^2 + V^2 + W^2}{2}\right) = \kappa_m^2 \frac{M}{R} - \frac{1}{R}\left(\kappa_m^2 M^2 + \kappa_u^2 U^2 + \kappa_v^2 V^2 + \kappa_w^2 W^2\right)
\tag{38}
$$

from which we see that the kinetic energy of the flow decays due to viscous dissipation with the only energy input coming from the forcing of the mean shear, as should be. This energy equation also shows that there is no blow-up since we have linear energy input with quadratic dissipation and any non-laminar statistically steady state must have $0 < M < 1$.

Brief analysis of the model

An analysis of the SSP model (37) is given in [4] and also in the notes for lecture 1. Although the laminar flow is stable for *all* Reynolds numbers, as is the case for the full linearized Navier-Stokes PDEs, the SSP model (37) allows the onset of non-trivial steady states for sufficiently large R. For those non-trivial states,

1. V redistribute M to create U. This is the $\sigma_u\, MV$ term, and $-\sigma_u UV$ is the corresponding 'Reynolds stress'.

2. the streaks U are unstable leading to the growth of W. This is the $\sigma_w\, UW$ term, this is an instability, not a direct forcing. The corresponding 'Reynolds stress' on U is the $-\sigma_w W^2$ term,
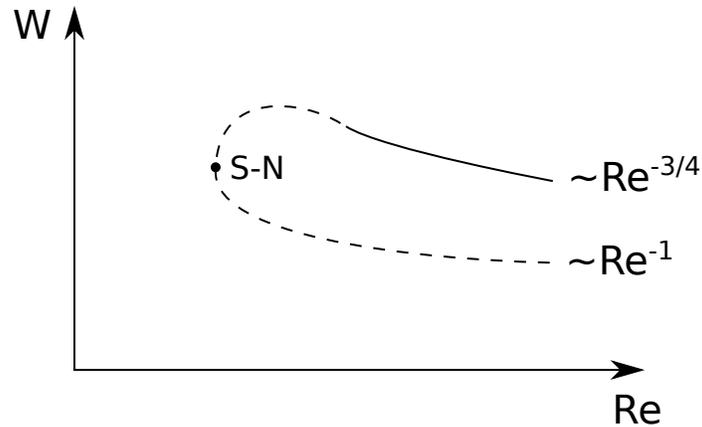
Figure 7: Non-trivial steady states of the SSP model (37), Re is the Reynolds number. The dashed line indicates instability of the solution. The laminar flow at $W = 0$ is stable for all Re.

3. Ah, Ha! *nonlinear feedback* from W onto V, the $\sigma_v W^2$,

4. but also a feedback onto M, $\sigma_m W^2$, pushing up the transition threshold.

This process supports onset of non-trivial steady states for R above a critical value and these states arise from 'saddle-node' bifurcations, although typically *both states are unstable from onset*. Because of the higher dimensionality, both the saddle and the 'node' are unstable at onset. In the low order model, the upper branch of solutions (the node), gains stability at higher R. The bifurcation and scaling of W are sketched in Figure 7. The lower branch saddle scales like $W \sim R^{-1}$ and the upper branch like $W \sim R^{-3/4}$.

References

[1] J. MOEHLIS, H. FAISST, AND B. ECKHARDT, *A low-dimensional model for turbulent shear flows*, New Journal of Physics, 6 (2004), pp. 56+17.

[2] F. WALEFFE, *Hydrodynamic stability and turbulence: Beyond transients to a self-sustaining process*, Studies in Applied Mathematics, 95 (1995), pp. 319–343.

[3] F. WALEFFE, *Transition in shear flows. nonlinear normality versus non-normal linearity*, Physics of Fluids, 7 (1995), pp. 3060–3066.

[4] F. WALEFFE, *On a self-sustaining process in shear flows*, Physics of Fluids, 9 (1997), pp. 883–900.