1. Show that a skew-symmetric operator has pure imaginary eigenvalues and orthogonal eigenvectors. Give an example of a skew-symmetric matrix and of a skew-symmetric differential operator.

2. Let (x, y) denote the innner product of vectors x and y. What is the minimum of $F = (x, Ax)/2 - (b, x)$ where $A = [3, 2; 1, 5]$ and $b = [1, 2]'$?

3. For the same A as above, what is the maximum and the minimum of (x, Ax) subject to the constraint $(x, x) = 1$?

4. If A is an N-by-N symmetric matrix with eigenvectors x_i and corresponding eigenvalues λ_i, write the solution of $Ax = b$ in terms of the eigenvectors and eigenvalues.

5. Verify that $u_{mn}(x, y) = \sin(m\pi x/L) \sin(n\pi y/H)$ are the eigenfunctions of the Laplacian in the rectangle $[0, L] \times [0, H]$ with zero boundary conditions on the sides and find the eigenvalues. Specify m and n. Write the solution to Poisson's equation $\nabla^2 u = f(x, y)$ with zero boundary conditions on the sides of the same rectangle in terms of these eigenfunctions and eigenvalues.

6. A solid body consists of a rectangular plate of length a and width b together with a rod of length c attached to a corner of the plate and perpendicular to it. The mass of the plate is M_p and the mass of the rod is M_r. Both are homogeneous. Find the center of gravity of the body. Find the principal moments and axes of inertia of the body if $a = 2m$, $b = 1m$, $c = 1m$ and $M_p = 3kg$, $M_r = 1kg$. (For simplicity, assume that the plate and rod are infinitely thin).

7. Consider the eigenvalue problem $(1 - x^2)\Phi'' - 2x\Phi' = -\lambda\Phi$ for the function $\Phi(x)$ in the domain $-1 \le x \le 1$ with eigenvalue λ. (a) Show that $\lambda > 0$ if Φ and Φ' are bounded at ± 1. (b) Use a change of variable $r = a(x+1)$ for some a to get rid of λ. (c) Discuss the behavior of $\Phi(r)$ as $r \to 0$.

8. Solve $z^5 = 1 + i$. Plot all solutions in the complex plane.

9. Strang 4.4.15: Show that the equation of a circle $|z - z_0|^2 = R^2$ can be rewritten as

$$pz\bar{z} + qz + \overline{qz} + r = 0 \qquad (p,\ r\ \text{ real}). \tag{1}$$

Substitute $w = 1/z$ and show that this gives a similar equation for a circle in the w-plane. Thus inversion maps circles into circles.

10. Consider the triangular region inside $z = 0, i, 1 + i$. What is the image of this region under the mapping (a) $w = z^2$, (b) $w = e^z$, (c) $w = 1/(z - i)$, (d) $w = \ln z$?

11. Find solution(s) of Laplace's equation in a wedge of angle α that have zero boundary conditions on the sides of the wedge. What can you say about the regularity of the solutions at the tip of the wedge as a function of α? (i.e. how do the solutions and their derivatives behave near the tip of the wedge? For $\alpha = \pi$, solutions are just multiples of $u(x, y) = y$ and perfectly regular as $x, y \to 0$, but this is not the case for other α's).

12. We derived that 2D, incompressible, irrotational fluid flow around a cylinder is given by the complex potential

$$F(z) = U\left(z + \frac{1}{z}\right) - i\frac{\Gamma}{2\pi}\ln z \tag{2}$$

where U is the x-component of velocity at infinity and Γ is the circulation.

(a) Show that the velocity components (u, v) are given by

$$u - iv = \frac{dF}{dz} \tag{3}$$

(b) Use the the Joukowsky mapping and a rotation to find the complex potential for flow about a plate of length L inclined at an angle α with the flow direction. [Hint: Start from the inclined plate and rotate by α to get the plate parallel to the x-axis then use the Joukowsky mapping $z + c^2/z$ to map to a circle. Be careful that the flow at infinity is modified by the rotation, so the potential you will need is not exactly (2).]

(c) Find Γ so the velocity at the trailing edge of the plate is finite. [Hint: the velocity is given by dF/dz. Use the chain rule $dF/dz = (dF/dz_1)/(dz/dz_1)$ to find the velocity implicitly. This avoids some algebra.]

13. Use contour integration to compute the integral

$$\int_0^{2\pi} \frac{\cos 4\theta}{4 + \cos\theta} d\theta.$$