**1.** Consider the ODE $\ddot{x} + \mu(\dot{x}^3 - \dot{x}) + x = 0$.
*(a) To what type of mechanical system does this ODE correspond?*
*(b) What type of solutions do you expect as $t \to \infty$?*
*(c) Find an explicit expression for the long time solution in the limit $\mu \ll 1$. Your solution must be accurate to order $\mu$ and valid for time scales $t \gg 1/\mu$.*
*(d) What is the discrete version of this ODE using Euler's method?*

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(a) Oscillator, amplified if $\mu(\dot{x}^3 - \dot{x}) < 0$ i.e. $|\dot{x}| < 1$; damped if $\mu(\dot{x}^3 - \dot{x}) > 0$ i.e. $|\dot{x}| > 1$;
(b) Expect periodic solutions because of (a).
(c) Try perturbation expansion $x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots$, substitute in equation and match like powers of $\mu$:

$$(\mu^0): \quad \ddot{x}_0 + x_0 = 0 \longrightarrow x_0(t) = Ae^{it} + c.c.$$

$$(\mu^1): \quad \ddot{x}_1 + x_1 = \dot{x}_0 - \dot{x}_0^3$$

$$= iAe^{it} - i^3A^3e^{i3t} - 3(iA)^2(-iA^*)e^{it} + c.c. = iA\left(1 - 3|A|^2\right)e^{it} + iA^3e^{i3t} + c.c.$$

So there will be a resonance at order $\mu$, unless $|A| = 1/\sqrt{3}$. In which case $x_1(t) = -iA^3/8e^{i3t} + c.c.$. Thus

$$x(t) = \frac{1}{\sqrt{3}}e^{it} - \frac{i\mu}{24\sqrt{3}}e^{i3t} + c.c. + O(\mu^2) = \frac{2}{\sqrt{3}}\cos t + \frac{\mu}{12\sqrt{3}}\sin 3t + O(\mu^2).$$

(d) Rewrite as a first order system: $y_1 = x$, $y_2 = \dot{x}$ then

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1 + \mu(y_2 - y_2^3),$$

so Euler's method will read

$$y_1^{n+1} = y_1^n + hy_2^n, \quad y_2^{n+1} = y_2^n + h\left[\mu(y_2^n - (y_2^n)^3) - y_1^n\right]$$

where $y_i^n$ is an approximation to $y_i$ after $n$ time steps and $h$ is the time step.

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**2.** *Solve $u_t = Du_{xx}$ with initial conditions $u(x,0) = 100$ and boundary conditions $u(0,t) = u(L,t) = 0$.*

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$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-Dn^2\pi^2 t/L^2} \sin n\pi x/L$$

where

$$u(x,0) = 100 = \sum_{n=1}^{\infty} A_n \sin n\pi x/L$$

so by orthogonality of the $\sin n\pi x/L$ in $[0,L]$:

$$A_n = \frac{2}{L}\int_0^L 100 \sin n\pi x/L \, dx = \frac{200}{n\pi}(1 - \cos n\pi) = \frac{200}{n\pi}(1 - (-1)^n).$$

Note that the $A_n$ go to zero as $n \to \infty$ only like $1/n$, pretty slow convergence (need lots of terms) however as soon as $t > 0$ the exponential saves the day and the series expansion for $u(x,t)$ converges faster and faster as time increases.

**3.** *How big a time step (approximately) could you use to solve the ODE $y' = f(t, y)$ with an explicit scheme such as Euler or RK2 for the following $f(t, y)$:*
*(a)* $f(t, y) = -y^3$
*(b)* $f(t, y) = \sin(1000y)$
*(c)* $y = \begin{pmatrix} y1 \\ y2 \end{pmatrix}$ *and* $f(t, y) = \begin{pmatrix} -1 & -10^{-3} \\ 10^6 & -10^3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

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(a) $\dot{y} = -y^3$ clearly converges toward zero. $df/dy = -3y^2$, so by linear stability analysis we need to choose a time step $h$ s.t. $|1 - 3y^2 h| < 1$, i.e. $h < 2/(3y_0^2)$.
(b) $|df/dy| = 1000|\cos y| \leq 1000$ so $h < 2/1000$.
(c) Eigenvalues: $\lambda = \left[-(10^3 + 1) \pm \sqrt{(10^3 + 1)^2 - 8\,10^3}\right]/2 \approx \left[-(10^3 + 1) \pm (10^3 + 3)^2\right]/2 = -2, -10^3$ so $h < 2/1000$.

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**4.** *In the following, $t$ is time and $x$ is a spatial variable.*
*(a) To what well-posed physical problems do the PDES $u_t = u_{xx}$ and $u_{tt} = u_{xx}$ correspond?*
*(b) Do $u_{ttt} = u_{xx}$ and $u_{tttt} = u_{xx}$ correspond to well-posed physical problems?*
*(c) What type of physics (e.g. diffusion, wave propagation, ...) correspond to the equations $u_{tt} = -u_{xxxx}$ and $u_t = 3u_x + u_{xxx}$?*

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(a) Heat (diffusion) equation and wave equation, respectively.
(b) Look for $u(x, t) = e^{i(kx - \omega t)}$ then $u_{ttt} = u_{xx} \Rightarrow (-i\omega)^3 = -k^2$ or $-i\omega = (-k^2)^{1/3} = -|k|^{2/3}, |k|^{2/3} e^{\pm i\pi/3}$. The last two roots have positive real parts proportional to $|k|^{2/3}$ but $|k|$ is unbounded (Fourier wavenumber) so these correspond to exponential growth with unbounded growth rate. Makes no physical sense. Solutions will not be continuously dependent on initial conditions, a small change of sufficiently high wavenumber will make a huge difference. Ill-posed.
Same sort of business for $u_{tttt} = u_{xx}$. Moral: time derivative cannot be higher than spatial derivative.
(c) $u_{tt} = -u_{xxxx}$, Look for $u(x, t) = e^{i(kx - \omega t)}$ then $(-i\omega)^2 = -(ik)^4$ or $\omega = \pm k^2$. Waves, dispersive (speed depends on wavenumber $k$). This is the *beam* equation (vibrations of a beam).
$u_t = 3u_x + u_{xxx}$ so $-i\omega = 3ik - ik^3$ or $\omega = k^3 - 3k$. Dispersive waves again. This is the linearized KdV equation.