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Proposal for a Self-Sustaining Mechanism in Shear Flows

with an interpretation for the streak spacing

Fabian Waleffe

1 Introduction

Experiments at low to moderate Reynolds number have consistently shown that the near wall region of shear flows exhibits strong spanwise variation in the downstream velocity (streaks). The spanwise spacing of these streaks always turns out to be very close to about 100 in wall units ($\frac{\nu}{u_*}$ where $u_* = (\nu du/dy)^{\frac{1}{2}}_{wall}$). At higher Reynolds number there is some controversy as to whether or not these streaks still exist, or are at all significant if they do. Various explanations have been advanced, most often based on linear theory. Although linear theory shows how large streaks can be created, it is unable to predict why the spacing 100+ should be picked out. There is simply no selection mechanism provided by linear behavior. Thus the argument that some fast non-linear mechanism provides broad-band disturbances which are then sheared according to linear theory does not work to explain the well-defined streak spacing.

Here a different explanation is advanced. It is proposed that the streak spacing of 100+ should be considered as a critical Reynolds number for transition from a laminar 1D flow to a 3D finite amplitude state in shear flows. The 100+ spacing would then correspond to the smallest Reynolds number at which a flow can be *maintained* different from the laminar flow. This idea is supported by the minimum channel simulations of Jimenez and Moin, as shown in the next section. A complete mechanistic picture of the bifurcated flow is proposed in the later sections and has been proposed before.

It is believed that this number is also relevant to fully turbulent flows. As one approaches the wall the effective Reynolds number is reduced and the flow eventually undegoes a "transition" from turbulent to laminar. The point at which this occurs would again be determined by the critical Reynolds number, 100+. It would represent the smallest *active* scale of a turbulent flow.

In fact, the proposal is that a key element in our understanding of turbulence in shear flows is to capture the *self-sustaining* 3D non-linear mechanism responsible for the bifurcation of laminar shear flows. Couette flow is the example of choice, because no other instability mechanism (2D, linear) seem to obscure the issue.

2 100+ or 1000? Same concept

The minimum channel computations clearly indicate that whether a "turbulent" flow can be computed or not, critically depends on the spanwise width of the computational periodic box being larger than about 100+. Below that number the flow returns to a laminar state. Jimenez and Moin show that this is the case for 3 different Reynolds numbers: 2000, 3000 and 5000. These are Reynolds number measured on the half height of the box, h. But when the spanwise width correspond to 100+, the box is much narrower than it is high. It seems that it would be more appropriate to measure the Reynolds number using the width of the box rather than the height. The transformation formula is :

$$Re_z = Re \frac{\lambda_z}{2h}$$

where Re_z is the Reynolds number based on the centerline velocity and the *half spanwise width*, while $Re = U_c h/\nu$ is based on the center-line velocity and the *half-height*. Using the information given in their fig. 4, one checks that when the flow ceases to be laminar, if

- $Re = 5000$ then $Re_z \simeq 1200$
- $Re = 3000$ then $Re_z \simeq 1200$
- $Re = 2000$ then $Re_z \simeq 1000$

In other words, Re_z has the much quoted value of about 1000 for transition to "turbulence" in Poiseuille flow !

In the following sections, a complete self-sustaining mechanism is proposed. One essential element in that process is the formation of large streaks by downstream rolls such that $\frac{\partial}{\partial z}u = O(\frac{\partial}{\partial y}u)$, followed by the subsequent 3D instability of that 2D flow. This process would be able to sustain itself provided the Reynolds numbers based on the friction velocity and the vorticity thicknesses (in the y and z direction) be larger than about 40 or 50, i.e.:

$$R_*^c \simeq 40$$

Thus the laminar flow would bifurcate to a 3D non-linear state when:

$$\frac{u_* \delta}{\nu} \; and \; \frac{u_* \zeta}{\nu} > 50$$

where δ and ζ are the "vorticity thicknesses" in the y and z directions, respectively. For a parallel shear flow δ and ζ are given by the size of the biggest downstream roll fitting in a vorticity layer. Examples are given below. In other words, the "magic" number 100+

is simply another version of the critical Reynolds number for transition to "turbulence". $Re_*^c \simeq 45$ could be a more universal measure for transition in shear flows, whether one studies a Couette flow, Poiseuille flow, B.L, M.L, etc...

- In the case of Couette flow:

$$u = U_w \frac{y}{h}$$

the friction velocity and the vorticity thickness are given by:

$$u_* = \sqrt{\frac{\nu U_w}{h}}, \quad \delta = 2h$$

Thus

$$\frac{u_* \delta}{\nu} = 2\sqrt{\frac{U_w h}{\nu}} = \sqrt{\frac{(2U_w)(2h)}{\nu}}$$

and thus $R_*^c \simeq 40$ is equivalent to $R^c \simeq 1600$ or $R^c \simeq 400$, depending on how the Reynolds number is defined. The periodic box should be at least as wide as $4h$ to observe a "bifurcated" flow at those Reynolds numbers.

- For a Poiseuille flow:

$$u = U_c(1 - \frac{y^2}{h^2})$$

u_* is given by:

$$u_* = \sqrt{2}\sqrt{\frac{\nu U_c}{h}}$$

while $\delta = h$, the half-channel height. So,

$$\frac{u_* \delta}{\nu} = \sqrt{2}\sqrt{\frac{U_c h}{\nu}} = \sqrt{2}\sqrt{Re}$$

and

$$R_*^c \simeq 40 \Leftrightarrow Re^c \simeq 800$$

The periodic computational box should be at least as wide as about $2h$ to observe a bifurcated flow.

- For a boundary layer, using the profile employed by Tollmien as a rough indication (cf. Drazin & Reid pg. 224), one gets:

$$u_* = \sqrt{1.7\frac{\nu U_\infty}{L}}$$

and using the full-boundary layer thickness as the length scale,

$$\frac{U_\infty L}{\nu} = \frac{1}{1.7}(\frac{u_* L}{\nu})^2$$

Measuring the Reynolds number on the displacement thickness, as Tollmien did, we need to multiply this by $\delta_*/L = 0.3416$. Thus $R_*^c \simeq 40 \Leftrightarrow Re^c \simeq 320$, while Tollmien found $Re^c \simeq 420$ for the *2D linear* instability. Thus the 2 mechanisms could be expected to occur at about the same Reynolds number.

- For a mixing layer, $u = U \tanh(y/h)$

$$u_* = \sqrt{\frac{\nu U}{h}}$$

Taking $\delta = 2h$:

$$R_* = \frac{u_* \delta}{\nu} = \sqrt{2}\sqrt{\frac{Uh}{\nu}} = \sqrt{2}\sqrt{R}$$

Thus $R_*^c \simeq 40 \Leftrightarrow R^c \simeq 800$, while the 2D, linear inflectional instability has a critical Reynolds number essentially equal to zero.

3 Streak formation

Consider the usual flow between 2 parallel plates. If x, y, z are, respectively, the downstream, normal to the plates and spanwise directions, the Navier-Stokes equations for a motion independent of the downstream x coordinate read:

$$\frac{\partial}{\partial t}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u = -\frac{\partial}{\partial x}p + \nu\nabla^2 u \quad (1)$$

$$\frac{\partial}{\partial t}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v = -\frac{\partial}{\partial y}p + \nu\nabla^2 v \quad (2)$$

$$\frac{\partial}{\partial t}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w = -\frac{\partial}{\partial z}p + \nu\nabla^2 w \quad (3)$$

where $\frac{\partial}{\partial x}p$ is a constant ($= 0$ for Couette flow), $u = u(y, z)$, $v = v(y, z)$, $w = w(y, z)$. The continuity equation is simply:

$$\frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0 \quad (4)$$

The equation for the u velocity component decouples and is linear. Once the v, w motion is determined, u can be computed independently. With the boundary conditions:

$$u(\pm 1) = \pm U_{wall}, \quad v(\pm 1) = w(\pm 1) = 0$$

the v, w motion corresponds essentially to viscously decaying downstream rolls. The x momentum is simply advected by the downstream rolls. Thus if one starts with a profile

$u = U(y)$ the momentum will be redistributed by advection and a spanwise variation with $\frac{\partial}{\partial z}u = O(\frac{\partial}{\partial y}u)$ will be induced.

"Over"-linearization around $U(y)$, such that $u = U(y) + u'(y,z)$, $v = v'(y,z)$, $w = w'(y,z)$ leads to, neglecting viscosity:

$$\frac{\partial}{\partial t}u' = -v'\frac{\partial}{\partial y}U$$

while the v, w motion is steady. This equation shows that u' grows linearly with time, but there is no spanwise selection. The spanwise structure is just that of v. Of course, the linear growth in time does not go on forever. It is simply the manifestation of x-momentum being advected away from a vertical plane into a horizontal plane. This advection would keep on going and the momentum would be carried back to a vertical plane through the term $w'\frac{\partial}{\partial z}u'$. The time scale of the streak formation is the time scale of the downstream roll, *not* $\frac{\partial}{\partial y}U$. It is the opinion of this writer that this mechanism is the only "algebraic" growth of interest.

It was checked numerically that an amplitude of the roll of the order Re^{-1} induces streaks with an amplitude of order 1. This leads to a quasi-steady state with momentum being convected from the upper wall to the lower wall and being transfered there by viscosity. For amplitude of the roll larger then about Re^{-1}, the momentum is being carried back up to the upper wall without having time to diffuse. Such a situation wouldn't correspond to the marginal self-sustained state that is sought here.

4 Why should we care about streaks?

It the previous section it was shown that inserting downstream rolls, even quite weak, in a shear flow, induces large streaks. However in the absence of any x-variation the downstream rolls will decay viscously, and the x momentum which is just being advected around will eventually go back to the original laminar flow. No "non-linear effects" will modify this picture.

However, it is quite likely that, at some time in its evolution, the flow will be quite sensitive to some disturbances and some instabilities might develop. In particular, the streaks introduce significant spanwise (z) inflections. One expect these shear layers to roll up creating vortices. This stage would be similar to a mixing layer or wake type instability. But one expects that due to the presence of the perpendicular shear ($\frac{\partial}{\partial y}u$), the instability and subsequent roll-up will occur for *oblique* waves. In fact, the growth rate might be significantly bigger than the conventional inflectional instability. Summing up this stage of the evolution, one expects the inflections in z of the $u(y,z)$ profile to be unstable to oblique disturbances ($u'(y,z)exp(i\alpha x)$), i.e. for which the vorticity is tilted in the x, y plane, possibly at about 45 deg. to benefit from the maximum stretching associated with the $\frac{\partial}{\partial y}u$ shear.

The instability would then go through a roll-up, creating vortices which would feed back on the original downstream rolls. Thus completing the loop for a self-sustaining 3D non-linear mechanism. The steps of this mechanism are as follows (fig.1):

- downstream rolls (v, w) induce large spanwise variations in the u-velocity (streaks).
- an inflectional-type instability develops due to these spanwise modulations and leads to the formation of vortices, likely to be tilted downstream.
- these vortices feedback on the original downstream vortices to maintain them and the process is reinitiated.

5 3D Instability of a Spanwise Varying Mean Flow

As shown above, it is no mystery how to induce large (O(1)) spanwise variation in the u-component of velocity from small (O(Re^{-1})) vertical motions. The critical phase in the process is to see whether or not anything interesting happens beyond that. This corresponds to studying the instability of a flow with spanwise variation. For Couette flow an appropriate quasi-steady basic state could be:

$$U(y,z,[t]) = y + a\cos\frac{\pi}{2}y\ \cos\frac{\pi}{2}z\ e^{-\frac{\pi^2}{2R}t}$$

(if the appropriate weak downstream roll was included the decay rate of the streaks would be even smaller). a is an amplitude, an adequate value for it is $a = 0.5$. Linearizing around such a basic state, and after a few manipulations, one obtains the following equations for the vertical and spanwise velocity perturbations:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - R^{-1}\nabla^2\right)\nabla^2 v - \frac{\partial^2}{\partial y^2}U\frac{\partial}{\partial x}v + \frac{\partial^2}{\partial z^2}U\frac{\partial}{\partial x}v + 2\frac{\partial}{\partial z}U\frac{\partial^2}{\partial x\partial z}v = 2\frac{\partial}{\partial z}U\frac{\partial^2}{\partial x\partial y}w + 2\frac{\partial^2}{\partial y\partial z}U\frac{\partial}{\partial x}w \tag{5}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - R^{-1}\nabla^2\right)\nabla^2 w - \frac{\partial^2}{\partial z^2}U\frac{\partial}{\partial x}w + \frac{\partial^2}{\partial y^2}U\frac{\partial}{\partial x}w + 2\frac{\partial}{\partial y}U\frac{\partial^2}{\partial x\partial y}w = 2\frac{\partial}{\partial y}U\frac{\partial^2}{\partial x\partial z}v + 2\frac{\partial^2}{\partial y\partial z}U\frac{\partial}{\partial x}v \tag{6}$$

with boundary conditions:

$$v = \frac{\partial}{\partial y}v = w = 0\ at\ y = \pm 1$$

$$\frac{\partial^3}{\partial y^3} w = \frac{\partial^3}{\partial y^2 \partial z} v \; at \; y = \pm 1$$

These equations can be recognized as 2 coupled Orr-Sommerfeld equations. In the usual stability problem the equation for v decouples. Here, due to the spanwise variation of the basic state, there is a feedback from w (i.e. ω_y) to v. If the basic state is written as $U(y,z) = u_0(y) + u_1(y)\cos 2\beta z$, one expects 2 types of solution (fig.2):

- a subharmonic, e.g

$$v = e^{i\alpha x} \sum_n \hat{v}_n e^{i(2n+1)\beta z}$$

- a fundamental, e.g.

$$v = e^{i\alpha x} \sum_n \hat{v}_n e^{i2n\beta z}$$

with similar expressions for w.

6 Preliminary Results

Full simulations of Couette flow have been performed initiating "turbulence" with a large perturbation at a moderate Reynolds number (1600). The Reynolds number was then decreased to try to determine the critical Re. A non-laminar flow has been maintained for a time $T \simeq 2000$ at Re=400 (i.e. $R_* = 40$). Re=225, 256, 289 and 324 all decayed. The mechanism at play seems to correspond quite well to the fundamental instability conjectured above. A large streak is observed at the channel centerline, then a strong kink develops and some time later the channel is occupied by a single vortex inclined downstream. The evolution is however not yet clean enough to be fully conclusive. The numerical resolution was 16 X 33 X 16. John Kim has investigated Poiseuille flow by inserting a pair of oblique waves directly at a "marginal" Reynolds number. This type of initial condition favors the subharmonic instability described above. A non-trivial and surprisingly clean evolution is observed for some time at Re=850 and 900. It corresponds to an exchange between a $(0, 2\beta)$ mode and a pair of oblique modes $(\alpha, \pm\beta)$, much has the "subharmonic" route should proceed. Unfortunately, the flow eventually returns to laminar after a time of about $500h/<U>$. Imposing some symmetries on the solutions should help clarify the picture and distinguish between the various possible evolutions.

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References

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