Transition in shear flows: non-linear normality versus non-normal linearity

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Abstract

A critique is presented of recent works promoting the concept of non-normal operators and transient growth as the key to understanding transition to turbulence in shear flows. The focus is in particular on a simple model [Baggett *et al.*, *Phys. Fluids* **7** (1995)] illustrating that view. It is argued that the question of transition is really a question of existence and basin of attraction of nonlinear self-sustaining solutions that have little contact with the non-normal linear problem. An alternative nonlinear point of view [Hamilton *et al.*, *J. Fluid Mech.* **287** (1995)] that seeks to isolate a self-sustaining nonlinear process, and the critical Reynolds number below which it ceases to exist, is discussed and illustrated by a simple model. Connections with the Navier-Stokes equations and observations are highlighted throughout.

1 Introduction

The breakdown of laminar shear flows is a particularly devious problem. Linear stability analysis, as governed by the Orr-Sommerfeld equation, has shown that viscosity and no-slip boundary conditions may lead to instabilities, but it does not predict the observed bifurcations in simple flows, such as plane Couette and Poiseuille flows. A host of weakly nonlinear theories have also been motivated by that problem [1], but the most successful applications of those theories have not been to explain transition to turbulence in shear flows.

The consensus is that transition is a strongly nonlinear problem, but recent works have suggested instead that non-normality of the linearized operator could largely explain the phenomenon. This is because non-normality implies the potential for transient growth. The argument is that linear transient amplification of some disturbances could trigger non-linearities that would prevent the eventual viscous decay of those disturbances.

This is not a new idea. The fact that transient growth could somehow be responsible for the breakdown of laminar shear flows has been repeatedly proposed for several decades (e.g. [2]-[8]), and the most significant transient effect, which physically arises from the redistribution of streamwise momentum by motions normal to the shear, has been linked to the 3D character of transition, at least since the works of Benney [9, 10] and Stuart [11]. The same mechanism leads to the formation of "streaks" observed in the near-wall region of turbulent shear flows, but does not explain their characteristic spacing.

In 1981, Benney and Gustavsson [6] investigated what nonlinear effects could be triggered by transient growth. They sought to develop a complete nonlinear theory and identified several important restrictions about the class of nonlinear interactions that could be triggered by the transients. Part of their theory was applied to a turbulent boundary layer by Jang, Benney and Gran [12], to propose a mechanism for the generation of streamwise rolls. Those works were later criticized by Waleffe, Kim and Hamilton [13], who suggested an explicit mechanism based on transients that could lead to an even more powerful weakly non-linear theory, on a time scale $\epsilon^{-1/3}$ where ϵ measures the disturbance amplitude. The main conclusions of the work of Waleffe *et al.* however, are that nonlinear effects unrelated to the transients play a major role, at least at the moderate Reynolds numbers where transition is observed, and that the linear transient amplification does not significantly select a most amplified disturbance. In addition, the most amplified disturbances are streamwise independent and those can not trigger nonlinear effects that will prevent the eventual viscous decay (cf. Sect. 4).

Recent papers [14]-[25] on linear transients have been largely limited to detailed computations of maximum linear amplifications instead of seeking to explicitly link the linear transients to nonlinear effects. A contribution of the recent publications is to have incorporated the linear transients in the broader mathematical context of non-normal operators and shown a more systematic approach to quantifying the linear amplifications [20]. But from a physical point of view, little has been accomplished that establishes the relevance of linear transients to transition in shear flows.

Baggett, Driscoll and Trefethen (BDT) [25] have recently presented a simple model to illustrate how linear transients coupled to generic "nonlinear mixing" can lead to transition. A related, but somewhat more physical model has been discussed by Gebhardt and Grossmann [24]. Those models address the important question of nonlinear feedback, or "bootstrapping," which is critical in establishing the significance of the linear transients. Unfortunately, they are of dubious relevance to transition in shear flows and violate some basic properties of the Navier-Stokes equations, as discussed in Sect. 2.

One characteristic of the recent works on linear transients and non-normality is actually to minimize the importance of nonlinearity which is viewed as a "generic mixer" whose role is to "recycle linearly amplified outputs into inputs" [7, 24, 25]. That point of view strongly limits the ultimate predictive power of the theory and perhaps the only prediction that can come out of that, is a bound on the asymptotic scaling with Reynolds number of the smallest amplitude of disturbances that may cause transition [20, 25]. One can not expect more precise predictions about the onset of turbulence without detailed consideration of the nonlinear effects. In addition, the physical evidence is that the most significant nonlinear effects triggered by transients in the Navier-stokes equations seem mostly to reduce the non-normality and the associated transient growth by adjusting the mean flow (see discussion following Eq. 7) , and not to recycle outputs into inputs [13, 26].

Another characteristic of those recent works is a confusing mix of general statements about nonlinear systems with non-normal linearized operators, with the specific problem of transition to turbulence in shear flows. Some of the issues are clarified in Sect. 4, where it is shown that nonlinearity is especially important precisely because the linearized operator is non-normal.

In view of these criticisms and the earlier results [13], it appears that the main, and perhaps only, physically significant effect of the transients is that a strong ($O(1)$) spanwise modulation (i.e. "streaks") of the streamwise velocity can be induced by weak ($O(1/R)$) streamwise rolls. This suggests that, short of a complete non-linear theory, one can perhaps develop a weakly nonlinear approximation, in which the starting point is a linearization around a spanwise modulated flow

$U(y,z)$, instead of the laminar flow $U(y)$ [26, 27, 28].

There is, however, a multitude of such spanwise modulated flows $U(y,z)$ and they all suffer a slow viscous decay. Hence, the theory must include nonlinear effects that will both *sustain* and *select* the spanwise modulated flow $U(y,z)$. There is hope for a weakly nonlinear approximation because streamwise rolls of $O(1/R)$ are sufficient to sustain an $O(1)$ modulation of the streamwise velocity. All that is needed, then, is a streamwise dependent structure, also of $O(1/R)$, to maintain the rolls by its quadratic nonlinear self-interaction. The streamwise independent structure is itself sustained by the "instability" of the spanwise modulated flow $U(y,z)$. "Sustenance" is here meant in a statistical sense. It is not suggested that the resulting process should be steady, nor periodic.

Note that this approach is meant as an approximate theory at the moderate R typical of transition, not as an asymptotic theory as $R \to \infty$. The notation $O(1/R)$ indicates here that those particular motions are smaller by about a factor of R, and thus that a great deal of simplification might be achieved by discarding all but a few critical nonlinear interactions.

Instead of directly seeking to develop such an approximation, our approach [13, 29] has been to first acquire experimental evidence for the validity of the underlying picture. Starting from an equilibrated turbulent solution and tracking it down in domain size, we isolated a simple process in the Navier-Stokes equations in which the three elements of the approximation can be clearly recognized.

In section 3, a simple model is discussed that illustrates that process and the other issues mentioned above. The model has a "laminar" fixed point and the linearized operator around that laminar point is non-normal. However, the nonlinear effects triggered by the linear transient suppress the non-normality and the transient growth and do not recycle outputs into inputs. A new element, sustained by the instability of the "streaks," is introduced that sustains the "streamwise rolls," who themselves sustain the streaks by redistributing the "mean shear".

A suggestion for improvement of models based on the "proper orthogonal decomposition" is given in the discussion section.

2 Non-Normal Linearity: the BDT model and conjecture

The BDT model is

$$\dot{u} = Au + ||u||Bu \tag{1}$$

with $u(t)$ in R^3. A is a real, constant non-normal matrix of the form

$$A = \begin{pmatrix} -2/R & \beta & 0 \\ 0 & -2/R & \beta \\ 0 & 0 & -2/R \end{pmatrix} \tag{2}$$

with $R > 0$ representing the Reynolds number. B is any real skew-symmetric ($B^T = -B$) matrix. The skew-symmetry is imposed so that the non-linear term conserves energy. There are at least two major deficiencies with this model.

First, the form of the non-linearity is unphysical as it makes use of the global norm of the solution $||u|| = (u_1^2 + u_2^2 + u_3^2)^{1/2}$. In the Navier-Stokes equations (NSE), the BDT nonlinearity is equivalent to replacing the advective nonlinearity, $\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}$, by $||\mathbf{u}||\, \mathbf{c} \cdot \boldsymbol{\nabla}\mathbf{u}$ where $||\mathbf{u}|| = (\int_V \mathbf{u} \cdot \mathbf{u}\, dV)^{1/2}$ and $\mathbf{c}$ is an arbitrary divergence-free velocity field, independent of $\mathbf{u}$. The operator $\mathbf{c} \cdot \boldsymbol{\nabla}$ is then skew-symmetric with appropriate boundary conditions. That is clearly a major simplification of the advective non-linearity that leaves out most of the nonlinear physics. A more appropriate expression, inspired by the rotational form of the nonlinearity in the NSE, $\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla q=\mathbf{\Omega}\cdot\mathbf{u}+\nabla q$, with $\mathbf{\Omega}=\frac{1}{2}[(\nabla\mathbf{u})^T-\nabla\mathbf{u}]$, could be

$$B=\begin{pmatrix} 0 & u_1-u_2 & u_1-u_3 \\ u_2-u_1 & 0 & u_2-u_3 \\ u_3-u_1 & u_3-u_2 & 0 \end{pmatrix} \tag{3}$$

with $u=(u_1,u_2,u_3)^T$ and of course dropping the $||u||$ in Equ.(1). This form can be generalized to a whole class of nonlinearities by adding suitable projection constants where partial operators appear in the NSE (e.g. $\partial_x \to d_x$, etc...). A most disturbing aspect of the BDT model however is that any of those more realistic forms for the nonlinearity lead to uninteresting and unphysical solutions that simply grow for all times (along paths that almost annihilate the nonlinearity, e.g. $u_1=u_2=u_3$ for the nonlinearity (3)).

This leads to the *second* deficiency, that the form of the linear term is also unphysical as it corresponds to a *frozen mean* problem, thus violating energy conservation. The BDT model (1) is equivalent to decomposing the total velocity field $\mathbf{v}$ into a base flow $\mathbf{U}$ plus a perturbation $\mathbf{u}$, $\mathbf{v}=\mathbf{U}+\mathbf{u}$, and considering the equation for the perturbation. In the NSE this corresponds to the perturbation equation

$$\mathbf{u}_t+\mathbf{U}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{U}+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\nu\nabla^2\mathbf{u} \tag{4}$$

The problem with this decomposition is that it is not orthogonal, $\int_V \mathbf{U}\cdot\mathbf{u}\,dV\neq 0$, hence the energy is not simply the sum of the base flow energy $\int \mathbf{U}\cdot\mathbf{U}\,dV$ plus the perturbation energy $\int \mathbf{u}\cdot\mathbf{u}\,dV$. There is also an "interaction energy" $2\int \mathbf{U}\cdot\mathbf{u}\,dV$. Thus the *non-normality* of A, emphasized in BDT and companion papers, is *frozen-in*. This is well-known to researchers in turbulence and nonlinear stability theory, and the decomposition into a *mean* plus a perturbation is used to avoid that complication. Then one has orthogonality of the decomposition and the equations for the divergence-free fields have the form

$$\begin{aligned} \mathbf{u}_t+\mathbf{U}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{U}+(\mathbf{u}\cdot\nabla\mathbf{u}-\overline{\mathbf{u}\cdot\nabla\mathbf{u}}) &= -\nabla p+\nu\nabla^2\mathbf{u} \\ \mathbf{U}_t+\mathbf{U}\cdot\nabla\mathbf{U}+\overline{\mathbf{u}\cdot\nabla\mathbf{u}} &= -\nabla P+\nu\nabla^2\mathbf{U} \end{aligned} \tag{5}$$

The growing perturbation $\mathbf{u}$ will modify the mean flow $\mathbf{U}$, and thus the form of the matrix A in (1). This important effect is not included in the BDT model that corresponds to a "frozen mean" problem and thus does not conserve energy [24].

Energy input into the NSE can only come from the boundary conditions, pressure gradient or a body force. As a result the energy input is proportional to the velocity $\mathbf{u}$ not to the velocity squared as in the BDT model (1). This is saying that energy input into a realistic simple model should come from a forcing term independent of the velocity.

Model (1) is thus of a rather artificial class. Yet, its purpose is to promote the idea that transient linear amplification is the key to understanding transition in shear flows, and that the exact nature of the nonlinear terms is rather unimportant. The BDT models present much evidence for a conjecture [20, 25] that some disturbances scaling as $\epsilon=O(R^{-3})$ will trigger transition. The only arguments that enter that estimate are transient growth of $O(R)$ combined with generic quadratic nonlinearity. It should then be applicable to shear flows given the numerous verifications [8],[14]-[25], of transient growth of $O(R)$. However, BDT emphasize that their conjecture for shear flows is only that $\epsilon=O(R^{\alpha})$ for $\alpha<-1$. To be of any interest, such a criterion must give a result sharper than the obvious $\epsilon = O(R^{-1})$ obtained by balancing nonlinear terms $O(\epsilon^2)$ with viscous decay terms of $O(\epsilon/R)$ [30]. BDT mention that numerical evidence seems to indicate amplitude scalings of $R^{-5/4}$ for Couette flow and $R^{-7/4}$ for Poiseuille flow. The rationalization [20, 25] for the discrepancy with the R^{-3} is that nonlinearity in the NSE "acts across modes via selection rules that the simple equations (1) do not model." In other words, transition in shear flows is not as simple as linear transient growth with generic quadratic nonlinearity after all [31].

Of course these estimates are asymptotic as $R \to \infty$, and one wonders then about the relevance of a criterion such as $\epsilon = O(R^{-7/4})$ as $R \to \infty$ in Poiseuille flow which is linearly unstable [1] at $R = 5772$, has nonlinear subcritical bifurcations [1] down to $R = 2900$ and evidence [32] for a different class of localized subcritical nonlinear solutions down to $R = 2330$. The linear instability may even prevent the possibility of testing such a criterion in Poiseuille flow. For Couette flow, one wonders if the experimental evidence is accurate enough to distinguish between $R^{-5/4}$ and R^{-1}, but in any case that scaling is so much closer to the trivial R^{-1} than to the predicted R^{-3} that nonlinearity is significantly more important than advocated by BDT.

The question of transition *from the laminar flow* reduces essentially to an initial value problem whose long time *statistical* behavior is highly sensitive to the initial conditions. The linearization around the laminar state is unable to predict which initial conditions will lead to transition. For instance, the largest transient growth is achieved for purely streamwise disturbances, but it can be proven that all such disturbances eventually decay. Although the question of the asymptotic scaling as $R \to \infty$ of the smallest perturbations that will trigger transition is not without merit, it is the opinion of this author that the main issue is to identify and characterize the nonlinear self-sustaining solutions that arise above a critical finite Reynolds number. Experimental and numerical observations, as well as some theoretical results, indicate that there is a *critical Reynolds number* R_c, below which any initial condition eventually settles onto the laminar solution, and above which other asymptotic states are possible. Those new solutions are and remain $O(1)$ away from the laminar solution as $R \to \infty$. Higher into the turbulent regime, the observations show signs of a coherent process in the near-wall region.

3 Nonlinear model of self-sustained flows

Recent advances on the nonlinear bifurcation to turbulence in shear flows include the theoretical foundations for a complete nonlinear process based on a spanwise varying mean flow by Benney [27], the nonlinear steady solutions discovered by Nagata [33] in Couette flow, and the surprisingly simple self-sustaining process isolated by this author in collaboration with John Kim and Jim Hamilton [13, 26, 29]. In the latter works, the approach has been to come *from the turbulent side* by progressively reducing the Reynolds number to isolate the basic nonlinear mechanism responsible for maintaining turbulence, and hence for finite amplitude transition. Transition is really a question of existence and basin of attraction of *other* self-sustained nonlinear solutions, that do not necessarily have any connection with the laminar solution (see e.g. Nagata [33] and Fig.1).

The simple process isolated through direct numerical simulations [13, 29] and analysis [26] consists of three elements: (**1**) Streamwise rolls redistribute the mean momentum to sustain a large $O(1)$ spanwise (z) modulation (called "streaks") in the streamwise (x) velocity. The rolls need only be $O(1/R)$. (**2**) The spanwise varying mean flow breaks down through an instability arising from the spanwise inflections in the flow. (**3**) The primary nonlinear development of the instability is to sustain the streamwise rolls.

A simple model of this observed process has been proposed earlier [26] and is further discussed hereafter. The model consists of 4 nonlinear equations

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \\ m \end{pmatrix} = \frac{1}{R}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{pmatrix} - \frac{1}{R}\begin{pmatrix} \lambda\, u \\ \mu\, v \\ \nu\, w \\ \sigma\, m \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\gamma\, w & v \\ 0 & 0 & \delta\, w & 0 \\ \gamma\, w & -\delta\, w & 0 & 0 \\ -v & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \\ m \end{pmatrix} \tag{6}$$

The variables u, v, w, m have a direct connection to the self-sustaining process isolated in the full NSE. The first component u represents the amplitude of the spanwise modulation of the streamwise velocity $[U(y, z, t) - \bar{U}]$, v is the amplitude of the streamwise rolls that consists mostly of vertical velocity $[V(y, z, t)]$ in the NSE (and a spanwise component $W(y, z, t)$ by continuity), w is the amplitude of the inflectional streak instability that consists mostly of spanwise velocity $[\exp(i\alpha x)w(y, z, t) + c.c.]$, and m represents the amplitude of the mean shear $[\partial\bar{U}(y, t)/\partial y]$ where the average is over horizontal planes x, z. The constants $\lambda, \mu, \nu, \sigma$ are positive and represent viscous decay rates, while γ and δ are nonlinear interaction coefficients that should have the same sign.

Note that the nonlinearity is skew-symmetric and quadratic; the mean field m is not frozen, it is modified by the growing perturbation ($-uv$ term in the m equation); the forcing is constant (the σ/R term forcing the mean flow m). The forcing represents the effect of the imposed pressure gradient in Poiseuille flow or the imposed velocity of the walls in Couette flow. The total energy evolves according to $d/dt(u^2 + v^2 + w^2 + m^2)/2 = \sigma m/R - (\lambda u^2 + \mu v^2 + \nu w^2 + \sigma m^2)/R$, and the negative definite quadratic term on the right-hand side dominates at large amplitudes, hence there is no unbounded growth. The total energy decays unless $0 < m < 1$.

The γuw term in the w equation will induce exponential growth of w, thus representing the streak instability. The δw^2 term in the v equation then represents the feedback on the streamwise rolls from nonlinear self-distortion of the streak instability. Those constants γ, δ should depend indirectly on the Reynolds number, through adjustment of the mode structures, in a more realistic truncated model of the self-sustaining process. The mv term in the u equation is the redistribution of the mean flow by streamwise rolls to induce the streaks u, it represents the advection of the mean field by streamwise rolls, $V(y, z, t)\partial\bar{U}/\partial y$, in the NSE. The $-uv$ term in the m-equation represents the Reynolds stress, dominated by the streamwise rolls and streaks [29], in the equation for the mean

$$\frac{\partial\bar{U}}{\partial t} = \frac{1}{R}\frac{\partial^2\bar{U}}{\partial y^2} - \frac{\overline{\partial P}}{\partial x} - \frac{\partial}{\partial y}\overline{U(y, z, t)V(y, z, t)}. \tag{7}$$

The nonlinear model (6) admits the "laminar" state $u = v = w = 0$, $m = 1$, as a fixed point. Linearization around that state indeed leads to a non-normal matrix as emphasized by BDT and many other papers, but the transient growth of u from the vm term reduces the non-normality by reducing the mean m through the $-uv$ term [34]. Note that model (6) has no nonlinear term that "recycles the output u into input v". More interestingly there may be other non-trivial fixed points u_0, v_0, w_0, m_0 determined by the roots of the cubic

$$\mathcal{C}(X) \equiv \delta^4\, X^3 + \nu\delta^2\, X^2 + (\mu\gamma^2 + \lambda\delta^2 - \gamma\delta R)\, \sigma\, X + \lambda\nu\sigma = 0, \tag{8}$$

where $X = R^2 w_0^2/\mu$. The intercept at $X = 0$ is positive and equal to $\lambda\nu\sigma$, the slope there is the coefficient of the X-term and must be negative to have positive roots. Thus a necessary but not sufficient condition to have a root is that $R > (\mu\gamma^2 + \lambda\delta^2)/(\gamma\delta)$, which gives a lower bound on the critical Reynolds number. The exact critical Reynolds number is determined by

$$R_c = \min_{X \geq 0} \frac{\delta^4 X^3 + \nu\delta^2 X^2 + (\mu\gamma^2 + \lambda\delta^2)\,\sigma\, X + \lambda\nu\sigma}{\gamma\delta\sigma\, X} \tag{9}$$

(another approach is to find $X(R)$ such that $d\mathcal{C}/dX = 0$, then R_c from $\mathcal{C}(X(R)) = 0$).

For $R > R_c$, there are two roots that emerge from a *saddle-node bifurcation* as pictured in the bifurcation diagram below (Fig.1). It can be shown [26] that the smallest one is always an unstable "saddle-point" (one positive real eigenvalue, one negative real, two complex conjugates with negative real parts). The other solution is either an unstable node (two positive real eigenvalues, two complex conjugates with negative real parts) that quickly turns into un unstable spiral, or a stable node (two negative real eigenvalues, two complex conjugates with negative real parts) that becomes unstable, through a supercritical Hopf bifurcation. As R is increased, the limit cycle collides with the saddle point (homoclinic bifurcation) and most initial conditions eventually end up at the laminar fixed point that seems to be the only stable attractor. At higher R another stable limit cycle of much larger amplitude emerges, then later disappears, apparently through homoclinic bifurcations. At very large R the steady solution regains stability. These are typical results for a limited region of parameter space. Some specific numerical values are given below.

The periodic solution mimics the self-sustaining process isolated in [13, 26, 29], while the steady solutions correspond to Nagata's solutions [33]. The very small, or even non-existent, range of stability of the steady state solutions is particularly pertinent to the Nagata steady solutions that seem to exist at Reynolds numbers (≈ 150) significantly lower than observed experimentally (≈ 350), and were shown to be unstable by Clever and Busse [35]. The simple nonlinear model (6), provides an illustration of that surprising behavior.

Note that the laminar solution $(0, 0, 0, 1)$ remains stable for all values of R. Its basin of attraction includes the entire hyperplane $w = 0$. The asymptotic scaling as $R \to \infty$ of the two branches of nonlinear steady solutions is easily deduced from (8). The unstable lower branch behaves as $W = O(R^{-3/2})$, $V = O(R^{-2})$, $U = O(R^{-1})$ and $M \sim 1$. The (rarely) stable upper branch gives $W = O(R^{-3/4})$, $V = O(R^{-1/2})$, $U = O(R^{-1/2})$ and $M \to 0$.

The parameter values chosen for the figures ($\lambda = \mu = \sigma = 10$, $\nu = 15$) are inspired by the corresponding objects in the NSE. The decay rate ν is a bit larger because it corresponds to the only field that depends on all three coordinates x, y, z. λ and μ are the decay rates of the x-independent streaks and rolls. The mean has a decay rate $\sigma = 10 \simeq \pi^2$ because it is antisymmetric ($\sin \pi y$ mode). The nonlinear coupling coefficient δ was kept fixed at $\delta = 1$. The dynamical behavior has been explored as function of R and the nonlinear coefficient γ, linked to the growth rate of the streak instability [26].

The minimum R_c is achieved at $X = 7.154048$, for any γ. The upper branch is an unstable node (except at very large R) for $\gamma < 0.200269$ and a stable node above that. For $\gamma = 0.5$, the critical R is $R_c = 98.6325$ but the upper branch steady solution is stable only up to $R = 100.0232$ ($X = 8.398554$) after which a supercritical Hopf bifurcation takes place. The ensuing limit cycle is stable until $R = 101.0311$ ($X = 8.819635$) when it disappears in a homoclinic bifurcation by colliding with the lower unstable branch. Another stable limit cycle is observed in the approximate range $356 < R < 435$. It appears and disappears apparently through homoclinic bifurcations (Fig.2).

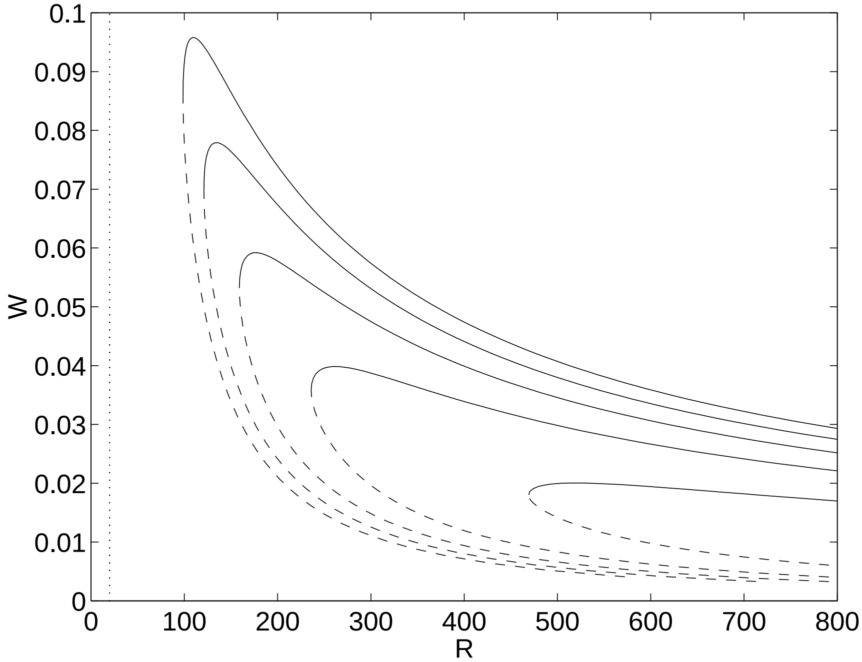

Fig. 1. Bifurcation diagram for the nonlinear model (6) of self-sustained flow, $\lambda = \mu = \sigma = 10$, $\nu = 15$, $\delta = 1$, $\gamma = 0.1, 0.2, \ldots, 0.5$ (outermost). The vertical dotted line is the absolute stability result $R_e = 20$. The dashed line is the unstable branch. The upper branch is not stable for all R (see Sect.3).

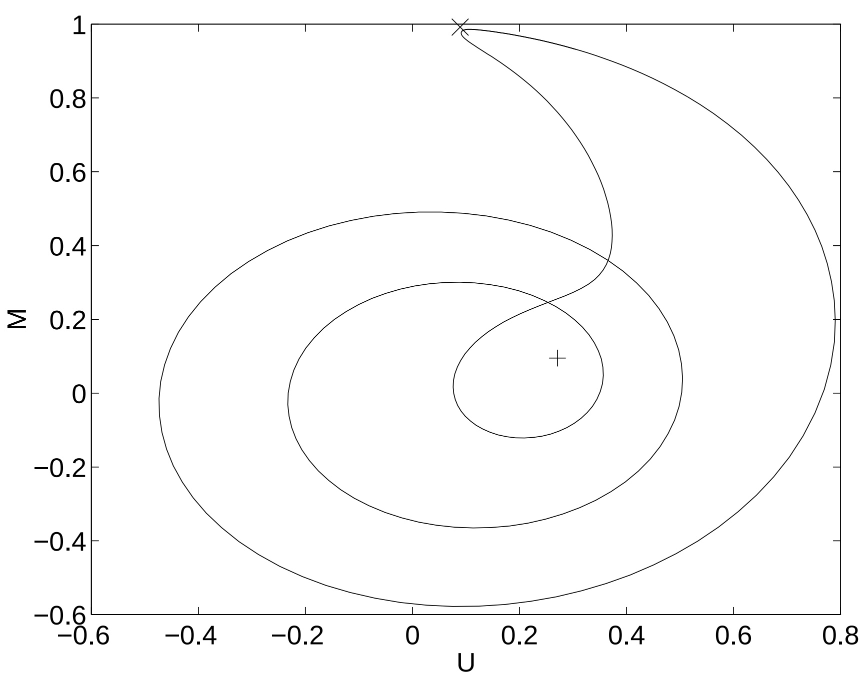

Fig. 2. The limit cycle at $R = 357$, $\gamma = 0.5$ projected onto the u, m plane with the upper branch (+) [0.2710, 0.0935,0.0512,0.0952] and the lower branch (x) [0.0891,0.0026,0.0085,0.9919] steady solutions.

4 Absolute stability and the energy integral

Multiplying the perturbation equation (4) by $\mathbf{u}$ and integrating over the whole domain, with periodic or homogeneous boundary conditions, leads to the following energy integral

$$\frac{1}{2}\frac{d}{dt}\int \mathbf{u}\cdot\mathbf{u}\,d\mathbf{x} = -\int \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}\,d\mathbf{x} - \frac{1}{R}\int \boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{u}\,d\mathbf{x}, \tag{10}$$

where $\mathbf{D} = [\boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^T]/2$ is the symmetric deformation-rate tensor of the laminar flow. The energy integral can be used to determine a Reynolds number R_e below which any perturbation decays and the laminar flow is thus *absolutely stable*. That Reynolds number is given by

$$R_e = \min_{\mathbf{u}} \frac{\int \boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{u}\,d\mathbf{x}}{-\int \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}\,d\mathbf{x}}\,. \tag{11}$$

The minimization is over all solenoidal fields $\mathbf{u}$ that satisfy the boundary conditions and have positive production $-\int \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u} \geq 0$. Note that the optimum field $\mathbf{u}$ is then a valid NSE initial condition for which the disturbance energy will grow initially provided $R > R_e$, as readily deduced from (10). Absolute stability for $R < R_e$ implies of course that transition can not occur unless R_e is finite. The only case in which R_e is not finite however, is when $\mathbf{D}$ is identically zero, hence when the basic state is one of pure solid body rotation. The problem in practice is that R_e is not only finite, but also much smaller than the observed critical Reynolds number for *sustained* perturbations.

Henningson and Reddy [36] assert that "linear mechanisms are necessary for transition." Their argument that "transition can not occur unless there is the potential for disturbance energy growth," is based only on the energy integral (10) and the number R_e. As reviewed in the previous paragraph, there is the "potential for disturbance energy growth" if and only if R_e is finite. Finite R_e does not imply anything about the importance of linear transient growth and non-normality. "Linear mechanisms" are "necessary" in the sense that the non-viscous *symmetric* component of the linearized operator, $\mathbf{D}$, should not vanish identically. The Henningson-Reddy statement is thus a rephrasal of the fact that transition can not occur if the basic state is one of pure solid body rotation.

To claim that linear transients and non-normality are necessary for transition in shear flows, in a substantial fashion, would require demonstrating that linear transients play an important role in the *nonlinear* dynamics, along the lines of the work by Benney and Gustavsson [6] and Waleffe, Kim and Hamilton [13]. This can certainly not be done through the energy integral (10) to which the nonlinear term and the skew-symmetric part of the linear operator do not contribute.

If the linearized operator is *normal*, and the nonlinear term conserves energy (or dissipates it), then linear mechanisms are actually *necessary and sufficient* for bifurcation from the laminar flow [36, 37]. This implies *supercritical bifurcation* under those conditions. That is because the Reynolds number R_e below which all perturbations decay, and the critical Reynolds number, R_l say, above which there is a linear exponential instability, coincide when the linearized operator is normal. R_l depends only on the linearized operator and R_e on its symmetric part. The most relevant fact for shear flows is that *linear mechanisms are insufficient for transition when the linearized operator is non-normal*, because then R_e is in general less than R_l. Linear theory is thus insufficient to predict transition, or lack thereof, in the range $R_e < R < R_l$. For plane Couette flow, this is a large gap as $R_e = 20.7$ and $R_l = \infty$. In summary, what is "necessary" is to consider nonlinear effects in the range $R_e < R < R_l$.

Consider the energy integral for model (6). Separating into the laminar base flow plus a perturbation, let $(u, v, w, m) = (0, 0, 0, 1) + (u, v, w, n)$. Substituting this expansion into the model (6) and taking the scalar product with (u, v, w, n) leads to the energy "integral"

$$\frac{1}{2}\frac{d}{dt}(u^2 + v^2 + w^2 + n^2) = -uv - \frac{1}{R}(\lambda u^2 + \mu v^2 + \nu w^2 + \sigma n^2) \tag{12}$$

The similarity between this and the energy integral (10) is striking, especially in the case of plane Couette flow $\mathbf{U} = (y, 0, 0)$ where the production $-\int \mathbf{u} \cdot \mathbf{D} \cdot \mathbf{u} = -\int uv \, d\mathbf{x}$. A lower bound on the critical Reynolds number for transition in model (6) is thus

$$R_e = \min_{\mathbf{u},\, uv<0} \frac{(\lambda u^2 + \mu v^2 + \nu w^2 + \sigma n^2)}{-uv} = 2\sqrt{\lambda\mu}. \tag{13}$$

It is clear that the minimum corresponds to $w = n = 0$, hence the energy integral does not lead to much information about the nature of the nonlinear self-sustaining solution that critically depends on $w \neq 0$. In fact it can be shown that *any* solution eventually decays if $w = 0$, for *any* finite R, not only for $R < R_e$, even though there can be substantial transient growth. The transient growth is largest when $w = 0$ actually. A similar statement, that streamwise independent flows — which lead to the largest transient growth — eventually return to the laminar state, can be proven for the NSE (by considering the energy integral for the spanwise motions v, w that decouple from the streamwise velocity component u and thus have no energy source).

Discussion

The identification and characterization of nonlinear solutions other than the laminar flow are the primary objectives in the quest to understand transition to turbulence in shear flows. This is especially true because the linearization around the laminar state leads to a non-normal operator, as clarified in the previous section. Non-normality of the linearized operator does not explain transition, but rather provides added motivation to abandon the linearization around the laminar state. As discussed in the introduction, a basic state where the streamwise velocity has a large spanwise modulation appears as a more fruitful state to "weakly non"-linearize about [27, 26].

The value of the simplistic model (6) discussed herein is primarily pedagogical, but one could consider a more realistic low-order model of the self-sustaining process by projection of the Navier-Stokes equations (NSE) onto the few modes suggested by the data [29] and the analyses [26, 29]. One simplification of the present model is that the streamwise dependent mode w does not feel the effect of the mean m, as it would in the NSE. That simplification seems valid for plane Couette flow for which the streak instability has purely real eigenvalues [26]. An obvious more general criticism is that turbulence is a high dimensional phenomenon that can not be appropriately described by low order models. However, the self-sustaining process isolated through numerical simulations [29] of the full Navier-Stokes equations is essentially low-dimensional.

One significant difference between the limit cycle of the 4th order model and the quasi-cyclic process isolated in the Navier-Stokes simulations is that energy is exchanged between the mean m and the other modes in the 4th order model while it is exchanged primarily between the streaks and x-dependent modes in the Navier-Stokes simulations ([29] fig.3). Although the energy is also introduced through the mean in the NSE, the mean flow remains remarkably constant throughout the quasi-cyclic process, with the mean shear concentrated in boundary layers near the wall. The streamwise rolls "scoop-up" slow, or fast, momentum from those boundary layers, thereby sustaining the streaks.

Other types of low order models, based on the *"proper orthogonal decomposition,"* have been proposed to describe some of the essential mechanisms of turbulent boundary layers ([38, 39] and references therein). The aim of those models differs from that of the present model which is to clarify the bifurcation to turbulence, but the self-sustaining process that has been identified at low Reynolds numbers is believed to remain a fundamental process in the near-wall region of high Reynolds number turbulent flows. One important aspect of the present model is to elucidate the origin of the streamwise rolls. The origin of the rolls has been outside the scope of the models based on the proper orthogonal decomposition.

In the latter models, the sustenance of the rolls has been artificially imposed by the expansion in known vector eigenfunction with dynamically determined amplitudes. The two degrees of freedom of the incompressible velocity field are kinematically linked in each vector function. The contribution of one vector function has the form $(u, v, w)^T = A(t)\mathbf{\Phi}(\mathbf{x})$ where $A(t)$ is the time-dependent amplitude and $\mathbf{\Phi}$ is a known vector function of position only. In the case of streamwise-independent disturbances for instance, the streamwise rolls v, w components are linked to the streamwise velocity u and the forcing of u from the redistribution of mean momentum by the rolls v, w artificially amplifies the rolls. This artificial feedback leads to exponential growth that is not present in the full equations. This holds for oblique disturbances also and could be remedied by separating the two kinematically independent degrees of freedom. If $\mathbf{\Phi}(y) \exp i(\alpha x + \gamma z)$ is an incompressible vector function for plane shear flow, it can be decomposed into two orthogonal parts $\mathbf{\Phi} = \mathbf{\Phi}_S + \mathbf{\Phi}_R$ where $\mathbf{\Phi}_S = P\mathbf{\Phi}$ and $\mathbf{\Phi}_R = (I - P)\mathbf{\Phi}$ with the projection matrix $P = \mathbf{p}\mathbf{p}^T/(\mathbf{p}^T\mathbf{p})$ and $\mathbf{p} = [-\gamma, 0, \alpha]$. In the expansion of the velocity field, the modes $\mathbf{\Phi}_S$ and $\mathbf{\Phi}_R$ should be assigned kinematically independent amplitudes. Hence, in this approach, the two degrees of freedom of the incompressible velocity field would remain while they were condensed into one in earlier models.

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[31] Actually, an explicit Navier-Stokes process that might lead to the R^{-3} scaling was proposed by Waleffe *et al.* [13], but evidence was also presented that at transitional R some nonlinear effects bypass the transients. If the process proposed there is indeed active, it must dominate at sufficiently large R, and lead to the R^{-3} scaling.

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