

Exact Solutions in Finite Compressible Elasticity via the Complementary Energy Function

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Abstract

The purpose of this paper is to establish a method of obtaining closed form solutions in isotropic hyperelasticity using the complementary energy, the Legendre transform of the strain energy function. Using the complementary energy, the stress becomes the independent variable and the strain the dependent variable. Some of the implications of this formulation of the equations are explored and illustrative examples of solutions for spherical and cylindrical inflation for several forms of the complementary energy are presented.

Introduction

The search for exact solutions of the equations of elasticity initially concentrated on incompressible materials. The simplified kinematics allows a wide variety of analytic solutions to be calculated. Much of the research concerned deformations possible in any material, the universal deformations, see for example Ogden [1]. Early on, though, Ericksen [2] showed that in compressible materials that the only universal solutions were homogeneous. The first inhomogeneous solutions for compressible materials were for the harmonic materials due to John [3]. Another large class of solutions, for strain energies depending on the invariants of the stretch tensor, were discovered by Carroll [4]. These included the harmonic materials as a special case. A third class that allows exact solutions are the Blatz-Ko materials [5]. A summary of all these solutions can be found in the papers by Horgan [6], Carroll [7] and Horgan [8]. A more systematic examination of the structure of the equations for spherical and cylindrical equations and their solution is given by Horgan and Murphy [10].

Treating the stress as the independent variable, rather than the strain or displacement, is a common tactic in linear elasticity problems. Examples include the Prandtl stress function in St. Venant torsion, the Airy function in plane strain or plane stress problems and the Love stress function in axisymmetric situations. The present paper utilizes this approach in finite elasticity. The first Piola-Kirchhoff stress is taken to be the independent variable. It is chosen so that the equilibrium equation is satisfied identically. The governing equations are obtained by requiring that the tensor obtained from differentiating the complementary energy corresponds to the gradient of a deformation field. Some constitutive restrictions on the complementary energy are derived from the requirement that the infinitesimal shear and bulk moduli be positive. Several classes of complementary energy are examined and solutions derived for spherical inflation.

The Legendre Transform

Let \mathcal{B} be a homogeneous three-dimensional continuum and let it occupy a fixed homogeneous configuration κ_0 at time $t = 0$. For any particle $X \in \mathcal{B}$, let \mathbf{X} be the position vector of X in the configuration κ_0 . Let \mathbf{x} be the position vector of X at time t , then a deformation of the body is given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t).$$

The deformation gradient \mathbf{F} is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial x_i}{\partial X_A} \mathbf{e}_i \otimes \mathbf{E}_A$$

where \mathbf{e}_i and \mathbf{E}_A are basis vectors in the current and reference configuration and the summation convention is assumed.

In the usual method of obtaining analytical solutions in hyperelasticity, one treats \mathbf{F} as the independent variable, where \mathbf{F} is the deformation gradient. The first Piola-Kirchhoff stress tensor \mathbf{P} is derived using the constitutive relation

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \tag{1}$$

where the strain-energy function $W(\mathbf{F})$ is a prescribed function of the deformation gradient. The equilibrium equation

$$\text{Div } \mathbf{P} = \mathbf{0} \tag{2}$$

then yields the differential equations of deformation.

In the alternative approach, utilized in this paper, one treats the stress \mathbf{P} as the independent variable. The domain of \mathbf{P} is the set of tensor fields satisfying (2). The Legendre transform of the strain-energy function $W(\mathbf{F})$ with respect to \mathbf{P} is denoted Ω :

$$\Omega(\mathbf{P}) = \mathbf{F} \cdot \mathbf{P} - W(\mathbf{F}) \quad (3)$$

The deformation gradient \mathbf{F} can then be derived using

$$\frac{\partial \Omega}{\partial \mathbf{P}} = \mathbf{F} + \frac{\partial \mathbf{F}}{\partial \mathbf{P}} \mathbf{P} - \frac{\partial W}{\partial \mathbf{P}} = \mathbf{F} + \frac{\partial \mathbf{F}}{\partial \mathbf{P}} \mathbf{P} - \frac{\partial \mathbf{F}}{\partial \mathbf{P}} \frac{\partial W}{\partial \mathbf{F}} = \mathbf{F}. \quad (4)$$

The differential equations of deformation are derived from the integrability conditions for \mathbf{F} . In cartesian coordinates, these are

$$\frac{\partial F_{iA}}{\partial X_B} = \frac{\partial F_{iB}}{\partial X_A} \quad i, A, B = 1, 2, 3. \quad (5)$$

In general these differential equations differ from those given by the usual method, and this alternative method can lead to new solutions.

Isotropy

Let $\mathbf{F} = \mathbf{R}\mathbf{U}$ be the right polar decomposition of \mathbf{F} . For isotropic materials, the strain-energy W can be written as a symmetric function W^\dagger of $\lambda_1, \lambda_2, \lambda_3$, the eigenvalues of \mathbf{U} , or equivalently as a function W^* of the principal invariants i_1, i_2, i_3 of \mathbf{U} :

$$W = W(\mathbf{F}) = W^\dagger(\lambda_1, \lambda_2, \lambda_3) = W^*(i_1, i_2, i_3), \quad (6)$$

where

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad i_3 = \lambda_1\lambda_2\lambda_3. \quad (7)$$

Define the Biot stress by

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \mathbf{U}}, \quad (8)$$

then if σ_i are the eigenvalues of $\boldsymbol{\sigma}$

$$\sigma_i = \frac{\partial W^\dagger}{\partial \lambda_i} \quad (9)$$

and (1) implies

$$\mathbf{P} = \mathbf{R}\boldsymbol{\sigma}. \quad (10)$$

Performing the Legendre transformation

$$\Omega = \sigma_1\lambda_1 + \sigma_2\lambda_2 + \sigma_3\lambda_3 - W. \quad (11)$$

The complementary energy Ω can be written as a symmetric function Ω^\dagger of $\sigma_1, \sigma_2, \sigma_3$, or equivalently as a function Ω^* of the invariants j_1, j_2, j_3 of $\boldsymbol{\sigma}$:

$$\Omega = \Omega(\mathbf{P}) = \Omega^\dagger(\sigma_1, \sigma_2, \sigma_3) = \Omega^*(j_1, j_2, j_3), \quad (12)$$

where

$$j_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad j_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \quad j_3 = \sigma_1\sigma_2\sigma_3. \quad (13)$$

The equation for \mathbf{F} ,

$$\mathbf{F} = \frac{\partial \Omega}{\partial \mathbf{P}} = \left(\frac{\partial \Omega^*}{\partial j_1} + j_1 \frac{\partial \Omega^*}{\partial j_2} \right) \mathbf{R} - \frac{\partial \Omega^*}{\partial j_2} \mathbf{P} + \frac{\partial \Omega^*}{\partial j_3} \text{adj}(\mathbf{P}), \quad (14)$$

where $\text{adj}(\mathbf{P})$ is the adjugate (i.e the matrix of cofactors) of \mathbf{P}

$$\text{adj}(\mathbf{P}) = e_{ijk} e_{ABC} P_{jB} P_{kC} \mathbf{e}_i \otimes \mathbf{E}_A \quad (15)$$

For (14), see for example Steigmann [9].

In the usual formulation of elasticity, the fact that $\det(\mathbf{F}) > 0$ implies that the rotation matrix in the polar decomposition for \mathbf{F} is uniquely determined. Since $\det(\mathbf{P})$ doesn't have a definite sign there are several different rotation matrices and accordingly different Biot stresses that can correspond to a given \mathbf{P} . For a thorough discussion of this issue see Ogden [1] page 358 ff.

Materials

We will consider the following forms of the complementary energy

$$\text{A} \quad \hat{\Omega}_A(j_1, j_2) = \frac{1}{2\mu} (f(j_1) - j_2), \quad (16)$$

$$\text{B} \quad \hat{\Omega}_B(j_1, j_2) = \frac{1}{2\mu} (g(aj_1 + j_1^2 - 4j_2) + bj_1), \quad (17)$$

$$\text{C} \quad \hat{\Omega}_C(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2\mu} (h((\sigma_1 + c)(\sigma_2 + c)(\sigma_3 + c)) + dj_1), \quad (18)$$

where a, b, c, d are constants and f, g, h are arbitrary functions of their arguments. In the strain energy formulation, the infinitesimal bulk and shear moduli are given by

$$3\kappa = \frac{\partial^2 W^\dagger}{\partial \lambda_1^2} + 2 \frac{\partial^2 W^\dagger}{\partial \lambda_1 \partial \lambda_2}, \quad 2\mu = \frac{\partial^2 W^\dagger}{\partial \lambda_1^2} - \frac{\partial^2 W^\dagger}{\partial \lambda_1 \partial \lambda_2} \quad (19)$$

(or any cyclic permutation of $\lambda_1, \lambda_2, \lambda_3$) where the derivatives are evaluated at $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Due to the properties of the Legendre transform, the equivalent expressions for the moduli in terms of the complementary energy are

$$\frac{1}{3\kappa} = \frac{\partial^2 \Omega^\dagger}{\partial \sigma_1^2} + 2 \frac{\partial^2 \Omega^\dagger}{\partial \sigma_1 \partial \sigma_2}, \quad \frac{1}{2\mu} = \frac{\partial^2 \Omega^\dagger}{\partial \sigma_1^2} - \frac{\partial^2 \Omega^\dagger}{\partial \sigma_1 \partial \sigma_2} \quad (20)$$

evaluated at zero stress. In addition the stretches at zero stress should be one. These conditions give the following restrictions on the complementary energy

Material A

$$f'(0) = 2\mu, \quad f''(0) = \frac{1}{3} \left(\frac{2\mu}{3\kappa} + 2 \right)$$

Material B

$$\frac{a}{4} + b = 2\mu, \quad g'(0) = \frac{1}{4}, \quad g''(0) = \frac{1}{3a^2} \left(\frac{2\mu}{3\kappa} + \frac{1}{2} \right)$$

Material C

$$d - c^2 = 2\mu, \quad h'(c^3) = -1, \quad h''(c^3) = \frac{1}{3c^4} \left(\frac{2\mu}{3\kappa} + 2 \right)$$

Spherical Inflation

Assume the deformation can be given in spherical polar coordinates by

$$r = \hat{r}(R), \quad \theta = \Theta, \quad \phi = \Phi. \quad (21)$$

The equilibrium equation (2) reduces to

$$\frac{d}{dR}(P_{rR}) + \frac{2}{R}(P_{rR} - P_{\theta\Theta}) = 0. \quad (22)$$

For this deformation field we have

$$\mathbf{R} = \mathbf{I}.$$

This means that the the principal values of the Biot stress are given by

$$\sigma_1 = P_{rR}, \quad \sigma_2 = \sigma_3 = P_{\theta\Theta}. \quad (23)$$

The constitutive equations give

$$F_{rR} = \frac{dr}{dR} = \frac{\partial\Omega}{\partial\sigma_1}, \quad F_{\theta\Theta} = \frac{r}{R} = \frac{\partial\Omega}{\partial\sigma_2}. \quad (24)$$

The integrability condition in this case is just

$$\frac{d}{dR}(RF_{\theta\Theta}) = F_{rR}, \quad (25)$$

which reduces to

$$R \frac{d\Omega_2}{dR} + \Omega_2 - \Omega_1 = 0, \quad (26)$$

where we have written

$$\Omega_1 = \frac{\partial\Omega}{\partial\sigma_1}, \quad \Omega_2 = \frac{\partial\Omega}{\partial\sigma_2}.$$

Material A

For model A,

$$\Omega_1 = \frac{1}{2\mu} (f'(j_1) - 2\sigma_2), \quad \Omega_2 = \frac{1}{2\mu} (f'(j_1) - (\sigma_1 + \sigma_2)). \quad (27)$$

Substituting into (26) immediately gives

$$R \frac{d}{dR} (f'(j_1) - (\sigma_1 + \sigma_2)) + (\sigma_2 - \sigma_1) = 0. \quad (28)$$

Using the equilibrium equation (22)

$$R \frac{d}{dR} \left(f'(j_1) - \frac{1}{2}j_1 \right) = 0. \quad (29)$$

This can be integrated to give

$$j_1 = \text{constant} = 3k_0. \quad (30)$$

Using the expression for j_1 and the equilibrium equation gives an equation for σ_1

$$R \frac{d\sigma_1}{dR} + 3\sigma_1 = 3k_0. \quad (31)$$

Integrating leads to

$$\sigma_1 = k_0 + \frac{k_1}{R^3}, \quad \sigma_2 = k_0 - \frac{k_1}{2R^3} \quad (32)$$

Now (24)₂ gives

$$r = \frac{R}{2\mu} \left(f'(3k_0) - 2k_0 - \frac{k_1}{2R^3} \right). \quad (33)$$

Material B

For model B, let

$$j_B = aj_1 + j_1^2 - 4j_2, \quad (34)$$

then

$$\Omega_1 = \frac{1}{2\mu} \left((a + 2\sigma_1 - 4\sigma_2)g'(j_B) + b \right), \quad (35)$$

$$\Omega_2 = \frac{1}{2\mu} \left((a + 2\sigma_1)g'(j_B) + b \right). \quad (36)$$

Substituting these into the compatibility equation (26) and using the equilibrium equation (26) leads to

$$(a + 2\sigma_1)R \frac{d}{dR} (g'(j_B)) = 0. \quad (37)$$

Thus, either

$$\sigma_1 = -\frac{a}{2} \quad \text{or} \quad j_B = \text{constant} = 3k_0. \quad (38)$$

The condition $j_B = 3k_0$ leads to the differential equation

$$(a - 2\sigma_1) \frac{d\sigma_1}{dR} + 3a\sigma_1 - 3\sigma_1^2 = 3k_0. \quad (39)$$

This has solution

$$\sigma_1 = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - k_0 + k_1 R^{-3}}. \quad (40)$$

Using (36),

$$r = \frac{R}{2\mu} \left(\left(2a + \pm \sqrt{\frac{a^2}{4} - k_0 + k_1 R^{-3}} \right) g'(3k_0) + b \right). \quad (41)$$

The other stress can be found using equilibrium and the condition (38) in the form

$$\sigma_2 = \frac{a}{2} \pm \left(\sqrt{\frac{a^2}{4} - k_0 + k_1 R^{-3}} - \frac{3R^3}{2\sqrt{\frac{a^2}{4} - k_0 + k_1 R^{-3}}} \right). \quad (42)$$

Material C

Using the form of stress energy given in (18) we obtain

$$\Omega_1 = \frac{1}{2\mu} ((\sigma_2 + c)(\sigma_3 + c)h'(j_C) + d), \quad (43)$$

$$\Omega_2 = \frac{1}{2\mu} ((\sigma_1 + c)(\sigma_3 + c)h'(j_C) + d), \quad (44)$$

where

$$j_C = (\sigma_1 + c)(\sigma_2 + c)(\sigma_3 + c).$$

The compatibility equation (26) becomes

$$R \frac{d}{dR} ((\sigma_1 + c)(\sigma_2 + c)h'(j_C)) + (\sigma_1 - \sigma_2)h'(j_C) = 0, \quad (45)$$

since

$$\sigma_2 + c = \sqrt{\frac{j_C}{\sigma_1 + c}}. \quad (46)$$

Equation (45) becomes

$$\sqrt{\sigma_1 + c} R \frac{d}{dR} (\sqrt{j_C} h'(j_C)) = 0, \quad (47)$$

which gives either $\sigma_1 = -c$ or

$$j_C = \text{constant} \quad j_C = k_0^2 \quad \text{or} \quad j_C = -k_0^2. \quad (48)$$

Thus

$$\sigma_2 + c = \frac{k_0}{\sqrt{\sigma_1 + c}} \quad \text{or} \quad \sigma_2 + c = \frac{k_0}{\sqrt{-(\sigma_1 + c)}}. \quad (49)$$

Using the equilibrium equation (22) gives

$$\frac{R}{2} \frac{d\sigma_1}{dR} + \sigma_1 + c = \frac{k_0}{\sqrt{\sigma_1 + c}} \quad \text{or} \quad \frac{R}{2} \frac{d\sigma_1}{dR} + \sigma_1 + c = \frac{k_0}{\sqrt{-(\sigma_1 + c)}} \quad (50)$$

with solutions

$$\sigma_1 + c = (k_0 + k_1 R^{-3})^{2/3} \quad \text{or} \quad \sigma_1 + c = -(-k_0 + k_1 R^{-3})^{2/3}, \quad (51)$$

where k_1 is an integration constant. From (45) the other stress component is

$$\sigma_2 + c = k_0(k_0 + k_1 R^{-3})^{-1/3} \quad \text{or} \quad \sigma_2 + c = k_0(-k_0 + k_1 R^{-3})^{-1/3}.$$

The displacement is obtained from (24)₂ as

$$r = \frac{R}{2\mu} \left(k_0(k_0 + k_1 R^{-3})^{1/3} h'(k_0^2) + d \right) \quad \text{or} \quad r = \frac{R}{2\mu} \left(-k_0(-k_0 + k_1 R^{-3})^{1/3} h'(-k_0^2) + d \right). \quad (52)$$

Cylindrical Inflation

Assume the deformation can be given in cylindrical polar coordinates by

$$r = \hat{r}(R), \quad \theta = \Theta, \quad z = \Lambda Z. \quad (53)$$

where Λ is a constant. The equilibrium equations (2) reduce to

$$\frac{d}{dR}(P_{rR}) + \frac{1}{R}(P_{rR} - P_{\theta\Theta}) = 0, \quad \frac{dP_{zZ}}{dZ} = 0. \quad (54)$$

For this deformation field we again have

$$\mathbf{R} = \mathbf{I}.$$

This means the the principal values of the Biot stress are given by

$$\sigma_1 = P_{rR}, \quad \sigma_2 = P_{\theta\Theta}, \quad \sigma_3 = P_{zZ}. \quad (55)$$

The constitutive equations give

$$F_{rR} = \frac{dr}{dR} = \frac{\partial\Omega}{\partial\sigma_1}, \quad F_{\theta\Theta} = \frac{r}{R} = \frac{\partial\Omega}{\partial\sigma_2}, \quad F_{zZ} = \Lambda = \frac{\partial\Omega}{\partial\sigma_3}. \quad (56)$$

The compatibility conditions in this case are

$$\frac{d}{dR}(RF_{\theta\Theta}) = F_{rR}, \quad \frac{d}{dR}(F_{zZ}) = 0, \quad (57)$$

which reduce to

$$R \frac{d\Omega_2}{dR} + \Omega_2 - \Omega_1 = 0, \quad \frac{d\Omega_3}{dR} = 0. \quad (58)$$

Material A

For model A,

$$\Omega_1 = \frac{1}{2\mu} (f'(j_1) - (\sigma_2 + \sigma_3)), \quad \Omega_2 = \frac{1}{2\mu} (f'(j_1) - (\sigma_1 + \sigma_3)), \quad \Omega_3 = \frac{1}{2\mu} (f'(j_1) - (\sigma_1 + \sigma_2)). \quad (59)$$

Substituting into (58) immediately gives

$$R \frac{d}{dR} (f'(j_1) - (\sigma_1 + \sigma_3)) + (\sigma_2 - \sigma_1) = 0. \quad (60)$$

Using the equilibrium equation (54)₁

$$R \frac{d}{dR} (f'(j_1) - \sigma_3) = 0. \quad (61)$$

This together with

$$2\mu\Omega_3 = 2\mu\Lambda = f'(j_1) - (\sigma_1 + \sigma_2), \quad (62)$$

implies that

$$j_1 = k_0, \quad \sigma_1 + \sigma_2 = f'(k_0) - 2\mu\Lambda, \quad \sigma_3 = k_0 + 2\mu\Lambda - f'(k_0). \quad (63)$$

Integrating (63)₂ using the equilibrium equation (54)₁ gives

$$\sigma_1 = \frac{1}{2} (f'(k_0) - 2\mu\Lambda) + \frac{k_1}{R^2}, \quad \sigma_2 = \frac{1}{2} (f'(k_0) - 2\mu\Lambda) + \frac{k_1}{R^2}. \quad (64)$$

Now (56)₂ gives

$$r = \frac{R}{2\mu} \left(\frac{3}{2} f'(k_0) - k_0 - \mu\Lambda - \frac{k_3}{R^2} \right). \quad (65)$$

Material B

For model B, let

$$j_B = aj_1 + j_1^2 - 4j_2, \quad (66)$$

then

$$\Omega_1 = \frac{1}{2\mu} \left((a + 2\sigma_1 - 2\sigma_2 - 2\sigma_3)g'(j_B) + b \right), \quad (67)$$

$$\Omega_2 = \frac{1}{2\mu} \left((a + 2\sigma_2 - 2\sigma_1 - 2\sigma_3)g'(j_B) + b \right), \quad (68)$$

$$\Omega_3 = \frac{1}{2\mu} \left((a + 2\sigma_3 - 2\sigma_1 - 2\sigma_2)g'(j_B) + b \right). \quad (69)$$

$$\Omega_3 = 2\mu\Lambda \quad \Rightarrow \quad 2(\sigma_2 - \sigma_3)g'(j_B) = (a - 2\sigma_1)g'(j_B) = b - 2\mu\Lambda. \quad (70)$$

Substituting this into the compatibility equation (58)₁ reduces to

$$R \frac{d}{dR} \left((2a - 4\sigma_1)g'(j_B) + b - 2\mu\Lambda \right) + 4(\sigma_2 - \sigma_1)g'(j_B) = 0. \quad (71)$$

Thus, either

$$\sigma_1 = \frac{a}{2} \quad \text{or} \quad j_B = \text{constant} = k_0. \quad (72)$$

The condition $\Omega_3 = 2\mu\Lambda$ leads to

$$\sigma_3 = \sigma_1 + \sigma_2 + K, \quad K = \frac{1}{2} \left(\frac{2\mu\Lambda - b}{g'(j_B)} - a \right) \quad (73)$$

Using this, the condition $j_B = k_0$, and the equilibrium equation gives the differential equation

$$(2a - 4\sigma_1)R \frac{d\sigma_1}{dR} + K^2 = 4a\sigma_1 - 4\sigma_1^2 + aK = k_0, \quad (74)$$

with solution

$$\sigma_1 = \frac{a}{2} \pm \sqrt{\frac{1}{4} (K^2 + aK + a^2 - k_0) + k_1 R^{-2}}. \quad (75)$$

The stress σ_2 can be found from the condition $j_B = k_0$,

$$\sigma_2 = \frac{a}{2} \pm \frac{K^2 + aK + a^2 - k_0}{4\sqrt{\frac{1}{4} (K^2 + aK + a^2 - k_0) + k_1 R^{-2}}}, \quad (76)$$

and σ_3 from (73)

$$\sigma_3 = a + K \pm \left(\sqrt{\frac{1}{4} (K^2 + aK + a^2 - k_0) + k_1 R^{-2}} + \frac{K^2 + aK + a^2 - k_0}{4\sqrt{\frac{1}{4} (K^2 + aK + a^2 - k_0) + k_1 R^{-2}}} \right). \quad (77)$$

The displacement is

$$r = \frac{R}{2\mu} \left(b - 2\mu\Lambda \mp \sqrt{\frac{1}{4} (K^2 + aK + a^2 - k_0) + k_1 R^{-2}} \right). \quad (78)$$

Material C

Using the form of stress energy given in (18) we obtain

$$\Omega_1 = \frac{1}{2\mu} ((\sigma_2 + c)(\sigma_3 + c)h'(j_C) + d), \quad (79)$$

$$\Omega_2 = \frac{1}{2\mu} ((\sigma_1 + c)(\sigma_3 + c)h'(j_C) + d), \quad (80)$$

$$\Omega_3 = \frac{1}{2\mu} ((\sigma_1 + c)(\sigma_2 + c)h'(j_C) + d). \quad (81)$$

where

$$j_C = (\sigma_1 + c)(\sigma_2 + c)(\sigma_3 + c).$$

The compatibility equation (58)₁ becomes

$$R \frac{d}{dR} ((\sigma_1 + c)(\sigma_3 + c)h'(j_C)) + (\sigma_1 - \sigma_2)(\sigma_3 + c)h'(j_C) = 0. \quad (82)$$

This gives

$$(\sigma_3 + c)h'(j_C) = \text{constant}. \quad (83)$$

The other compatibility equation $\Omega_3 = 2\mu\Lambda$ gives

$$(\sigma_1 + c)(\sigma_2 + c)h'(j_C) = 2\mu\Lambda. \quad (84)$$

Together (83) and (84) imply

$$j_C = k_0, \quad (\sigma_1 + c)(\sigma_2 + c) = \frac{2\mu\Lambda}{h'(k_0)}. \quad (85)$$

The second equation together with the equilibrium equation gives the differential equation

$$R \frac{d\sigma_1}{dR} + \sigma_1 + c = \frac{2\mu\Lambda}{(\sigma_1 + c)h'(k_0)}. \quad (86)$$

This has the solution

$$\sigma_1 = -c \pm \sqrt{\frac{2\mu\Lambda}{h'(k_0)} + k_1 R^{-2}}. \quad (87)$$

The other stresses are

$$\sigma_2 = -c \pm \frac{2\mu\Lambda}{h'(k_0)} \left(\frac{2\mu\Lambda}{h'(k_0)} + k_1 R^{-2} \right)^{-1/2}, \quad (88)$$

$$\sigma_3 = -c + \frac{k_0 h'(k_0)}{2\mu\Lambda}. \quad (89)$$

The displacement is obtained from (56)₂ as

$$r = \frac{R}{2\mu} \left(\pm \frac{k_0 (h'(k_0))^2}{2\mu\Lambda} \sqrt{\frac{2\mu\Lambda}{h'(k_0)} + k_1 R^{-2}} + d \right). \quad (90)$$

Conclusion

A new method for obtaining closed form solutions for spherical and cylindrical inflation of isotropic compressible materials is proposed in this paper. Exploiting the duality relation between the complementary energy and strain energy, taking the stress to be the independent variable, gives rise to a different set of equations than the usual approach using the strains. In some cases they are more tractable and allow new solutions to be found.

Postscript

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References

- [1] Ogden R.W., *Non-Linear Elastic Deformations*, Ellis Horwood, Chichester (1984).
- [2] Ericksen J.L., Deformations possible in every compressible isotropic perfectly elastic material, *J. Math. Physics* 1955 ; 24 : 126 – 128.
- [3] John F., Plane Strain Problems for a Perfectly Elastic Material of Harmonic Type, *Comm. Pure Appl. Math.* 1960 ; 13 : 239 – 296.
- [4] Carroll M.M., Finite Strain Solutions in Compressible Finite Elasticity, *J. Elasticity* 1988; 20 : 65 – 92.
- [5] Blatz P.J. and Ko. W.L., Application of finite elasticity to the deformation of rubbery materials, *Trans. Soc. Rheol.* 1960; 6 :223 – 251.
- [6] Horgan C.O., On axisymmetric solutions for compressible nonlinearly elastic solids, *Zeit. Angew. Math. Phys.* 1995; 46 : S107 – S125.
- [7] Carroll M.M., On obtaining closed form solutions for compressible nonlinearly elastic materials, *Zeit. Angew. Math. Phys.* 1995; 46 : S126 – S145.
- [8] Horgan C.O., Equilibrium solutions for compressible nonlinearly elastic materials, *Nonlinear Elasticity : Theory and Applications*, Fu Y.B. and Ogden R.W. eds, London Mathematical Society 2001; 135 – 159.
- [9] Steigmann D.J., Invariants of the Stretch Tensors and their Applications to Finite Elasticity Theory, *Mathematics and Mechanics of Solids*2002;7: 393–404
- [10] Horgan C.O. and Murphy J.G., A Lie Group Analysis of the Axisymmetric Equations of Finite Elastostatics for Compressible Materials, *Mathematics and Mechanics of Solids* 2005; 10: 311–333.