2.5 #2

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant, that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Ox) \qquad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Solution
Let $O = [o_{ij}]$. We compute:

$$D_i v(x) = \sum_{k=1}^{n} D_k u(Ox) o_{ki},$$
$$D_{ij} v(x) = \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) o_{ki} o_{lj}.$$

Since O is orthogonal, then $OO^T = I$ where I is the $n \times n$ identity matrix, thus for all $k, l = 1, \ldots, n$

$$\sum_{i=1}^{n} o_{ki} o_{li} = \delta_{kl} := \begin{cases} 1 & \text{if } k = l; \\ 0 & \text{if } k \neq l. \end{cases}$$

Thus

$$\begin{aligned} \Delta v(x) &= \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) o_{ki} o_{li} \\ &= \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) \left(\sum_{i=1}^{n} o_{ki} o_{li} \right) \\ &= \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) \delta_{kl} \\ &= \Delta u(Ox) = 0. \end{aligned}$$