

On the average number of octahedral modular forms

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Let $N > 0$ be an integer, χ a Dirichlet character modulo N , and k either 0 or 1. Let f be a primitive eigenform, not necessarily holomorphic, of level N and Nebentypus χ , and let $\lambda_f(n)$ be the eigenvalue of T_p on f . We say f is associated to a Galois representation $\rho : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{C})$ if

$$\begin{aligned}\lambda_f(p) &= \text{Tr}(\rho(\text{Frob}_p)) \\ \chi(p) &= \det(\rho(\text{Frob}_p))\end{aligned}$$

for all p not dividing N . Following [4], we define $S_{1/4,k}^{\text{Artin}}(N, \chi)$ to be the finite set of primitive weight k cuspidal eigenforms which admit an associated Galois representation. If $\rho : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{C})$ is a Galois representation, we define $\mathbb{P}\rho$ to be the composition of ρ with the natural projection $GL_2(\mathbb{C}) \rightarrow PGL_2(\mathbb{C})$.

Two-dimensional complex Galois representations fall naturally into four types; we call ρ dihedral, tetrahedral, octahedral, or icosahedral according as the projectivized image $\mathbb{P}\rho(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to a dihedral group, A_4 , S_4 , or A_5 . Cusp forms associated to Galois representations are classified likewise.

The latter three types are called “exotic”; it is widely believed that the number of exotic cusp forms of level N is at most N^ϵ . The first results in this direction are due to Duke [3]. These results were later sharpened by Wong [6] and Michel and Venkatesh [4]. The latter authors proved that

$$n^{\text{tetr}}(N, \chi, k) \ll_\epsilon N^{2/3+\epsilon}, n^{\text{oct}}(N, \chi, k) \ll_\epsilon N^{4/5+\epsilon}, n^{\text{icos}}(N, \chi, k) \ll_\epsilon N^{6/7+\epsilon}$$

where $n^T(N, \chi, k)$ is the number of weight k cusp forms of level N , Dirichlet character χ , and type T . (Note that $n^T(N, \chi, k) = 0$ unless χ sends complex conjugation to $(-1)^k$.)

The goal of this paper is to show that the Michel-Venkatesh bound on octahedral forms can be sharpened *on average* over squarefree levels N .

We begin by showing that, in the case of square-free level, one does not need to consider very many different Dirichlet characters χ when counting exotic cusp forms.

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Lemma 1. *Let N range over square-free integers. Then the number of Dirichlet characters χ of conductor N such that there exists an exotic cusp form of level N and Nebentypus χ is $O(N^\epsilon)$.*

Proof. Let ρ be the Galois representation attached to an exotic cusp form of level N and conductor χ . Let p be a prime dividing N . Then, since $p \parallel N$, the restriction $\rho : I_p \rightarrow GL_2(\mathbb{C})$ must decompose as $\chi \oplus 1$. In particular, the projection from $\rho(I_p)$ to $\mathbb{P}\rho(I_p)$ is an isomorphism. So $\chi(I_p)$ is a cyclic subgroup of either A_4, S_4 , or S_5 ; in particular, χ^{60} is unramified everywhere, whence trivial. Now the number of characters χ of level N such that $\chi^{60} = 1$ is $O(N^\epsilon)$, which proves the lemma. \square

Suppose from now on that N is squarefree. Let $n^{\text{Oct}}(N)$ be the number of octahedral cusp forms of level N . It follows from Lemma 1 and the theorem of Michel and Venkatesh that

$$n^{\text{Oct}}(N) \ll_\epsilon N^{4/5+\epsilon}.$$

Let $p : S_4 \rightarrow S_3$ be the natural surjection. Then every homomorphism $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow S_4$ can be composed with p to yield a homomorphism $\psi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow S_3$. By combining arguments from [6] and [4], we obtain the following sharpening of Theorem 10 of [6]:

Proposition 2. *Let k be 0 or 1, and N a positive squarefree integer. Let $\psi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow S_3$ be a homomorphism, and let $n_\psi^{\text{Oct}}(N)$ be the number of octahedral weight k cusp forms associated to Galois representations ρ such that $p \circ \mathbb{P}\rho = \psi$.*

Then $n_\psi^{\text{Oct}}(N) \ll_\epsilon N^{2/3+\epsilon}$.

Proof. Let χ be a character of level N . In the proof of [6, Thm. 10], Wong constructs an amplifier—that is, a set of complex numbers $\{c_n\}_{n \in \mathbb{N}}$ such that, for some absolute constants C and C' ,

- $\sum_{n \leq B} |c_n| \leq CB^{1/4}$.
- $\sum_{n \leq B} |c_n|^2 \leq CB^{1/4}$.
- If ρ is an octahedral Galois representation such that $p \circ \mathbb{P}\rho = \psi$, and f is a cusp form in $S_{1/4,k}^{\text{Artin}}(N, \chi)$ associated to ρ , then $|\sum_{n \leq B} c_n \lambda_f(n)| \geq C' B^{1/4} / \log B$.

In [4, §3], Michel and Venkatesh use the Petersson-Kuznetsov formula and standard bounds on Kloosterman sums to obtain the following inequality:

$$\sum_{f \in \Sigma} (f, f)^{-1} \left| \sum_{\substack{n \leq B \\ (n, N)=1}} c_n \lambda_f(n) \right|^2 \ll_\epsilon \sum_{\substack{n \leq B \\ (n, N)=1}} |c_n|^2 + (BN)^\epsilon B^{1/2} N^{-1} \left(\sum_{\substack{n \leq B \\ (n, N)=1}} |c_n| \right)^2 \quad (1)$$

where

- Σ is a set of eigenforms in $S_{1/4,k}^{\text{Artin}}(N, \chi)$;
- (f, f) is the Petersson self-product of f ;
- $\{c_n\}$ is an arbitrary sequence of complex numbers.

Speaking loosely, the idea of [4] and [3] is that, by the Petersson-Kuznetsov formula, the vectors $\{\lambda_f(n)\}_{f \in \Sigma}$ and $\{\lambda_f(m)\}_{f \in \Sigma}$ are “approximately orthogonal” when m and n are distinct integers. On the other hand, Wong shows that there exists a large set of n such that the Fourier coefficients $\lambda_f(n)$ are real numbers of a fixed sign as f ranges over Σ . The desired bound on Σ will follow from the tension between these two constraints.

In the above inequality, take $\{c_n\}$ to be Wong’s amplifier and Σ to be the set of octahedral forms in $S_{1/4,k}^{\text{Artin}}(N, \chi)$.

Note that $(f, f) = O(N \log^3 N)$ by [6, Lemma 6]. So the left hand side of (1) is bounded below by a constant multiple of

$$N^{-1} \log^{-3} N \sum_{f \in \Sigma} \left| \sum_{\substack{n \leq B \\ (n, N)=1}} c_n \lambda_f(n) \right|^2 \geq N^{-1} \log^{-3}(N) |\Sigma| (C')^2 B^{1/2} \log^{-2}(B)$$

while the right hand side is bounded above by a constant multiple of

$$B^{1/4} + (BN)^\epsilon B^{1/2} N^{-1} B^{1/2}.$$

Combining these bounds, one has

$$|\Sigma| \ll_\epsilon N \log^3(N) B^{-1/2} \log^2(B) (B^{1/4} + (BN)^\epsilon B N^{-1})$$

The bound is optimized when we take $B \sim N^{4/3}$, which yields

$$|\Sigma| \ll_\epsilon N^{2/3+\epsilon}.$$

Combined with the fact that the number of χ under consideration is $O(N^\epsilon)$, this yields the desired result. \square

Proposition 2, in combination with the theorem of Davenport and Heilbronn on cubic fields, allows us to improve Michel and Venkatesh’s bound on $n^{\text{Oct}}(N)$ in the average.

Theorem 3. *For all $\epsilon > 0$ there exists a constant C_ϵ such that*

$$(1/X) \sum_{\substack{N < X \\ N \text{ sq.free}}} n^{\text{Oct}}(N) < C_\epsilon N^{2/3+\epsilon}$$

for all $X > 1$.

Proof. Let f be an octahedral form of level N associated to a representation ρ . For each prime $p|N$, the group $\mathbb{P}\rho(I_p)$ is a cyclic subgroup of S_4 . (Recall that N is squarefree.) Define

- N_1 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is a nontrivial subgroup of the Klein four-group;
- N_2 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is contained in A_4 but not in the Klein four-group;
- N_3 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is not contained in A_4 .

Then $N_1 N_2 N_3 = N$. Let $\psi : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow S_3$ be the composition $p \circ \mathbb{P}\rho$. Then the fixed field L of $\ker \psi$ is a cyclic 3-cover of a quadratic number field K , where K/\mathbb{Q} is the unique quadratic field ramified precisely at primes dividing N_3 , and L/K is ramified only at primes dividing N_2 .

Let $b(N_2, N_3)$ be the number of such S_3 -extensions L . (For notational convenience we take $b(N_2, N_3)$ to be 0 when either N_2 or N_3 is not square-free.) From Proposition 2, we have

$$n^{\text{oct}}(N) \ll_{\epsilon} \sum_{\substack{N_1, N_2, N_3 \\ N_1 N_2 N_3 = N}} b(N_2, N_3) N^{2/3+\epsilon}. \quad (2)$$

Let T be the set of places of K dividing $3N_2\infty$, and let $G_T(K)$ be the Galois group of the maximal extension of K unramified away from T . Each cubic field counted in $b(N_2, N_3)$ is a cyclic 3-extension of K unramified away from T , so

$$b(N_2, N_3) \leq |\text{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z})|.$$

The Galois cohomology group above fits in an exact sequence

$$0 \rightarrow \text{Hom}(\text{Cl}_T(K), \mathbb{Z}/3\mathbb{Z}) \rightarrow \text{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z}) \rightarrow \prod_{v \in T} \text{Hom}(\text{Gal}(\bar{K}_v/K_v), \mathbb{Z}/3\mathbb{Z})$$

where $\text{Cl}_T(K)$ is the quotient of the class group of K by all primes in T . (See [5, (8.6.3)]). Let $h_3(N_3)$ be the order of the 3-torsion subgroup of the class group of K . Since $\dim_{\mathbb{F}_3} \text{Hom}(\text{Gal}(\bar{K}_v/K_v), \mathbb{Z}/3\mathbb{Z})$ is at most 4 (see [5, (7.3.9)]), we have

$$b(N_2, N_3) \leq h_3(N_3) 3^{4|T|} \ll_{\epsilon} N_2^{\epsilon} h_3(N_3). \quad (3)$$

Combining (3) and (2) yields

$$n^{\text{oct}}(N) \ll_{\epsilon} \sum_{\substack{N_1, N_2, N_3 \\ N_1 N_2 N_3 = N}} h_3(N_3) N^{2/3+\epsilon}.$$

Since the sums over N_1 and N_2 have length at most $d(N) = O(N^\epsilon)$, we have

$$n^{\text{oct}}(N) \ll_\epsilon \sum_{N_3|N} h_3(N_3) N^{2/3+\epsilon}.$$

So

$$\sum_{\substack{N < X \\ N \text{ sq.free}}} n^{\text{oct}}(N) \ll_\epsilon \sum_{N_3=0}^X h_3(N_3) \sum_{k=0}^{X/N_3} (kN_3)^{2/3+\epsilon} \leq X^{5/3+\epsilon} \sum_{N_3=0}^X h_3(N_3) (1/N_3). \quad (4)$$

The sum $\sum_{d=0}^X h_3(d)/d$ can be estimated as follows. Integration by parts yields

$$\sum_{d=0}^X h_3(d)/d = (1/X) \sum_{d=0}^X h_3(d) + \int_1^X \left(\sum_{d=1}^t h_3(d) \right) t^{-2} dt.$$

Now by the theorem of Davenport and Heilbronn [2, Theorem 3] we have $\sum_{d=0}^t h_3(d) = O(t)$. It follows that

$$\sum_{d=0}^X h_3(d)/d = O(\log X).$$

Substituting this bound into (4) gives

$$\sum_{\substack{N < X \\ N \text{ sq.free}}} n^{\text{oct}}(N) \ll_\epsilon X^{5/3+\epsilon}$$

which yields the desired result. □

Theorem 3 can be thought of as a bound for the number of quartic extensions of \mathbb{Q} whose Artin conductor, with respect to a certain 2-dimensional projective representation of S_4 , is bounded by X . This is quite different from the problem, recently solved by Bhargava [1], of counting the number of quartic extensions of \mathbb{Q} with *discriminant* less than X . For instance, quartic extensions attached to cusp forms of conductor N might have discriminant as large as N^3 .

References

- [1] Bhargava, M. Higher Composition Laws. Princeton Ph.D. thesis, 2001.
- [2] Davenport, H. and Heilbronn, H. On the density of discriminants of cubic fields. II. Proc. Roy. Soc. London Ser. A 322 (1971), no. 1551, 405–420.
- [3] Duke, W. The dimension of the space of cusp forms of weight one. Int. Math. Res. Notices 1995, no. 2, 99–109.

- [4] Michel, P. and Venkatesh, A. On the dimension of the space of cusp forms associated to 2-dimensional complex Galois representations. *Int. Math. Res. Not.* 2002, no. 38, 2021–2027.
- [5] Neukirch, J. and Schmidt, A. and Wingberg, K. *Cohomology of number fields.* Grundlehren der Mathematischen Wissenschaften, 323. Springer-Verlag, Berlin, 2000.
- [6] Wong, S. Automorphic forms on $GL(2)$ and the rank of class groups. *J. Reine Angew. Math.* 515 (1999), 125–153.