LECTURE 1

JE is away–no lecture 1!

LECTURE 2

LECTURE 3

I. Damped Chris and Pat

We spoke last time about Chris and Pat; recall that we modeled their troubled relationship via the differential equations

$$dp/dt = -c, dc/dt = p$$

or, more compactly

$$d\vec{\ell}/dt = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\ell}.$$

And we found that the solution to these differential equations with initial conditions

$$\vec{\ell}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

was

$$e\vec{l}l(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

Now suppose, to make this problem more realistic, we try to take into account the fact that feelings decay with time. That is, both $p(t)$ and $c(t)$ have a tendency to move towards 0. We can model this phenomenon by changing our differential equation to

$$dp/dt = -0.1p - c, dc/dt = p - 0.1c$$

or

$$d\vec{\ell}/dt = \begin{bmatrix} -0.1 & -1 \\ 1 & -0.1 \end{bmatrix} \vec{\ell}.$$

Adding a "factor of decay" to a model is often called "damping." This operation corresponds to adding a small negative multiple of I to our matrix B.

Let's keep the same initial conditions

$$\vec{\ell}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now we have

$$p_B(\lambda) = \det \begin{bmatrix} -0.1-\lambda & -1 \\ 1 & -0.1 - lambda \end{bmatrix} = (\lambda+0.1)^2 + 1$$

So the eigenvalues of B are

$$\lambda_1 = -0.1 + i, \lambda_2 = -0.1 - i.$$

The corresponding eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

and so once again we have

$$\vec{\ell}(0) = (1/2)\vec{v}_1 + (1/2)\vec{v}_2$$

. So

$$\vec{\ell}(t) = (1/2)e^{(-0.1+i)t}\vec{v}_1 + (1/2)e^{(-0.1-i)t}\vec{v}_2$$

Recall that

$$e^{-0.1+it} = e^{-0.1t}e^{it} = e^{-0.1}t(\cos t + i \sin t).$$

So we can factor out the $e^{-0.1t}$ and get

$$\vec{\ell}(t) = e^{-0.1t}[(1/2)e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + (1/2)e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}] = \begin{bmatrix} e^{-0.1t}\cos t \\ e^{-0.1}t \sin t \end{bmatrix}$$

Draw a picture of the trajectory; it's a spiral which cycles inwards towards the origin, which in this problem represents complete apathy.

Key remark: The relationship would spiral towards apathy *whatever the initial conditions were!* No matter how much love (or hate) is present at the start, all emotion eventually dissipates. This depressing conclusion leads us to the study of stability properties of differential equations.

II. Stability of differential equations

Let's recall the examples we've studied:

- $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: The solutions were of the form $c_1e^{3t}\vec{v}_1 + c_2e^t\vec{v}_2$. Draw this–exponential growth, converging on the direction of $\vec{v}_1$. (Here, $\vec{v}_1$ and $\vec{v}_2$ are the eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.) The eigenvalues of B are $3, 1$.

- $B = \begin{bmatrix} -0.1 & 1 \\ -1 & -0.1 \end{bmatrix}$ Here the system spiraled inward towards 0. The eigenvalues of B are $-0.1 - i, -0.1 + i$.

- $B = [\lambda]$. This is the ur-example

$$dy/dx = \lambda y.$$

  If $\lambda > 0$, exponential growth; if $\lambda < 0$, exponential decay. (If $\lambda = 0$, the situation is more delicate; ask, what is the solution in this case?)

From this, let's draw a general principle.

Theorem. Let

$$d\vec{y}/dx = B\vec{y}(*)$$

be a system of n linear differential equations. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of B.

Then

- If $\Re\lambda_i < 0$ for all i, all solutions to (*) decay to 0. This is called the *stable case.*

- If some $\Re\lambda_i > 0$, then "almost all" solutions to (*) grow exponentially. This is called the *unstable case.*

- if $\Re\lambda_i \leq 0$ for all i, and one or more of the $\Re\lambda_i$ is 0, then the equation is called *neutrally stable.* Typically, there will be either a nonzero stable state (as in $dy/dx = 0$ or a stable oscillation (as in the original Chris and Pat problem.)

Remark: The theorem above is true even if B is not diagonalizable. But be aware that you have no reason to believe this except that I tell you so–we haven't treated any non-diagonalizable cases.

III. Stability in the 2×2 case.

In case

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$p_B(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (\mathrm{Tr}B)\lambda + (detB).$$

So the eigenvalues are the roots of this polynomial, which by the quadratic formula are

$$\lambda = \frac{1}{2}(\mathrm{Tr}B \pm \sqrt{(\mathrm{Tr}B)^2 - 4\det B})$$

So OK. Draw the picture in the "trace-determinant plane" of the different regions. We observe that the eigenvalues are real if and only if $(\mathrm{Tr}B)^2 - 4\det B$ is positive. In particular, if $\det B < 0$, then the eigenvalues are real, and one is negative and one is positive–so fill in those two quadrants. If the eigenvalues are complex, they are complex conjugates

$$\lambda_1 = p + qi, \lambda_2 = p - qi$$

and we have

$$\Re\lambda_1 = \Re\lambda_2 = p = 1/2(\lambda_1 + \lambda_2) = 1/2\mathrm{Tr}B.$$

So stability or instability depends on whether the trace is positive or negative.

Finally, if the eigenvalues are real and the determinant is positive, then the eigenvalues are either both positive (if $\mathrm{Tr}B$ is positive) or both negative (if $\mathrm{Tr}B$ is negative.) This completes the diagram and gives us a complete answer, in terms of the entries of B, to the question: when is a 2×2 system of linear differential equations stable?

Give an example of negative determinant showing that not all initial conditions yield exponential growth; this is the reason for the "almost all" in the theorem stated above.

IV. Matrix exponentials

Return to the ur-example:

$$dy/dt = \lambda y$$

has the solution $y = e^{\lambda t}y(0)$.

Similarly, we would like to say

$$d\vec{y}/dt = B\vec{y}$$

has the solution

$$\vec{y} = e^{Bt}\vec{y}(0).$$

And it's true! The only problem is that we have to define the exponential of a matrix.

Theorem.

$$\vec{y} = e^{Bt}\vec{y}(0).$$

is the unique solution to

$$d\vec{y}/dt = B\vec{y}.$$

Outline of this:

- Give the power series definition.
- Observe that it's not obvious this converges; but it does.
- Show that it satisfies the equation.
- Show how it comes out if B is diagonalizable.
- Observe that the matrix exponential gives the right answer even if B is not diagonalizable; observe case of $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Magical!