

LECTURE 1

I. Column space, nullspace, solutions to the Basic Problem

Let A be a $m \times n$ matrix, and \vec{y} a vector in \mathbb{R}^m . Recall the fact:

Theorem: $A\vec{x} = \vec{y}$ has a solution if and only if \vec{y} is in the column space $R(A)$ of A .

Now let's prove a new theorem which talks about when such a solution is unique. Suppose \vec{x}, \vec{x}' are two solutions to $A\vec{x} = \vec{y}$. Then

$$A(\vec{x}' - \vec{x}) = A\vec{x}' - A\vec{x} = \vec{y} - \vec{y} = \vec{0}.$$

So $\vec{x}' - \vec{x}$ is in the *nullspace* $N(A)$ of A .

In fact, we can say more.

Theorem: Let \vec{x}_0 be a solution to $A\vec{x} = \vec{y}$. Then the set of all solutions is the set of vectors which can be written

$$\vec{x}_0 + \vec{n}$$

for some $\vec{n} \in N(A)$.

Proof omitted—it's not much more than the above argument.

We also get the corollary:

Theorem: Suppose \vec{y} is in the column space of A . $A\vec{x} = \vec{y}$ has a *unique* solution if and only if $N(A) = \vec{0}$.

Ex: Take

$$V = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Now

- $N(V)$ is the set $\vec{x} : x_1 + x_2 + x_3 = 0$

- $R(V)$ is the set $\text{span}\left\{ \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$.

In particular, the equation $V\vec{x} = \vec{y}$, if it has a solution, will never have a unique one. One minute contemplation: in view of the above theorem, how do we think of the set of solutions of

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}?$$

By contrast, take

$$D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 5 \end{bmatrix}.$$

Then **ask** what people think the nullspace and column space are.

- $N(D)$ is the zero space;
- $R(D) = \mathbb{R}^3$.

So, for every \vec{y} , the equation $D\vec{x} = \vec{y}$ has a unique solution (which we already knew, because D is invertible.)

II. Rank

Let's understand the averaging example a little more closely. If we try to solve

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

we get, after elimination,

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - y_1 \\ y_3 - y_1 \end{bmatrix}.$$

Provisional definition: The *rank* of a matrix A is the number of nonzero pivots in A after elimination.

So rank of V is 1, while rank of D is 3. (Elimination is already complete there!)

Let Z be the zero matrix. What is rank of Z ?

Now make a little table:

	V	D	Z
rank	1	3	0
nullspace	plane	$\vec{0}$	\mathbb{R}^3
column space	line	\mathbb{R}^3	$\vec{0}$

This may give us the idea that the rank tells us something about the “dimension” of the column space, which in turn seems to have something to do with the “dimension” of the nullspace. This will turn out to be exactly correct, once we work out the proper definition of the terms involved!

Note that we can now prove

Theorem (Invertibility theorem III) Suppose A is an $n \times n$ matrix such that $N(A) = \vec{0}$ and $\mathcal{R}(A) = \mathbb{R}^n$. Then A is invertible.

Proof. The equation $A\vec{x} = \vec{y}$ has a solution for every \vec{y} , because every \vec{y} is in the column space of A . This solution is always unique, because $N(A) = \vec{0}$. So $A\vec{x} = \vec{y}$ always has a unique solution. It now follows from invertibility theorem II that A is invertible.

LECTURE II

I. Linear independence and basis

Lots of important basic definitions today.

Remember our discussion from the first week:

Two vectors \vec{v}_1, \vec{v}_2 span a plane *unless* one is a scalar multiple of the other.

Today we'll talk about the proper generalization of the notion above to larger sets of vectors.

Definition: A set of vectors $\vec{v}_1, \dots, \vec{v}_n$ is *linearly independent* if and only if the only solution to

$$x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$$

is $x_1 = \dots = x_n = 0$.

Alternative 1: $\vec{v}_1, \dots, \vec{v}_n$ is linearly independent if and only if the equation

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} = \vec{0}$$

has only the solution

$$\begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} = \vec{0}.$$

Alternative 2, most compact of all: $\vec{v}_1, \dots, \vec{v}_n$ is linearly independent if and only if

$$N\left(\begin{bmatrix} | & | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix}\right) = \{\vec{0}\}.$$

So the notions of “linear independence” and “nullspace” are closely related.

Example: Two vectors \vec{v}_1, \vec{v}_2 . Suppose they are *not* linearly independent. Then there is an expression

$$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$$

such that x_1 and x_2 are *not* both 0. In other words, \vec{v}_1 and \vec{v}_2 are scalar multiples of each other.

So we can rephrase our fact from week 1:

Two vectors \vec{v}_1, \vec{v}_2 span a plane as long as they are linearly independent.

Now, a crucial definition.

Definition. Let V be a subspace of \mathbb{R}^m . A *basis* for V is a set of vectors $\vec{v}_1, \dots, \vec{v}_n$, which

- are linearly independent;
- span V .

For instance, let’s start our work by looking at the subspace of \mathbb{R}^2 :

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 0 \right\}$$

Ask: What are some vectors in this space? Can we produce a basis for it? Is it the only one?

Now (if time permits) turn to partners, and address the following question: can we think of a basis for \mathbb{R}^2 ? How many can we think of?

LECTURE III

Note: I won’t define row space and “left nullspace” this week.

At last we can define dimension.

Theorem. Let V be a subspace of \mathbb{R}^m . Then any two bases of V contain the same number of vectors. The number of vectors in a basis for V is called the *dimension* of V .

This is a wonderful theorem—if only I had time to prove it at the board. But I think the intuitive desire to believe the above fact is strong enough that I can get away without a formal proof.

Our goal for today will be to prove the conjectures from last time. Namely:

Theorem (Rank Theorem)

- The rank of A is the dimension of $\mathcal{R}(A)$.
- The dimension of $\mathcal{R}(A)$ plus the dimension of $N(A)$ is the number of columns of A .

Before we set off to prove this theorem, let me record some of the corollaries.

Corollary. The following are equivalent:

- $\text{rank}(A) = m$;
- $\mathcal{R}(A) = \mathbb{R}^m$;
- $A\vec{x} = \vec{y}$ has at least one solution for every \vec{y} .

Corollary. The following are equivalent:

- $\text{rank}(A) = n$;
- $N(A) = \vec{0}$;
- $A\vec{x} = \vec{y}$ has at most one solution for every \vec{y} .

Corollary. The following are equivalent:

- $\text{rank}(A) = m = n$;
- $A\vec{x} = \vec{y}$ has exactly one solution for every \vec{y} .
- A is invertible.

In other words, we have shown that an invertible matrix must be square!

So: now that we've eaten our dessert, let us turn to the vegetables—which in my opinion are actually quite tasty. We want to prove the theorem above.

FACT: Let A be an $m \times n$ matrix, and let B be an invertible $m \times n$ matrix. Then

1. $N(BA) = N(A)$.
2. $\mathcal{R}(BA)$ and $\mathcal{R}(A)$ have the same dimension.

Proof: Suppose \vec{v} is in $N(BA)$. Then $BA\vec{v} = \vec{0}$. Multiplying by B^{-1} , we see that $A\vec{v} = \vec{0}$. The other direction is obvious. This proves 1.

For part 2, let $\vec{v}_1, \dots, \vec{v}_d$ be a basis for $\mathcal{R}(A)$. So for every \vec{x} in \mathbb{R}^n , we have

$$A\vec{x} = k_1\vec{v}_1 + \dots + k_d\vec{v}_d.$$

Thus, we can write $BA\vec{x} = k_1Bv_1 + \dots + k_dBv_d$.

So $B\vec{v}_1, \dots, B\vec{v}_d$ span $\mathcal{R}(BA)$. Moreover, $B\vec{v}_1, \dots, B\vec{v}_d$ are linearly independent, because if

$$k_1B\vec{v}_1 + \dots + k_dB\vec{v}_d = 0$$

just multiply by B^{-1} and we see that all the k_i must be 0. So $\mathcal{R}(BA)$ has a basis with d elements, so has dimension d .

In particular, if A is any matrix, and U is the matrix resulting from Gaussian elimination, then

$$U = (E_1E_2E_3\dots)A$$

and we can take $B = (E_1E_2E_3\dots)$, which is invertible. So by the FACT above, we see that

- $N(A) = N(U)$
- $\dim\mathcal{R}(A) = \dim\mathcal{R}(U)$.

Thus, to prove the Rank Theorem, it suffices to prove it in case Gaussian elimination is already finished, since Gaussian elimination—as we have just seen—does not affect the numbers we are interested in.

There presumably won't be time to say much, if anything, more.