1 Opening stuff

- Pass out syllabus.
- Say name and office hours: (tentative) Monday, 3-4.
- Say I should be contacted by e-mail.
- Introduce Gergely.
- Emphasize gospel of homework. That success in this course, by which I mean learning the material, can *only* be gained by persistent attention to the homework problems. Remember that there is a great distance between understanding what I say in class and being able to work problems on your own. That the homework problems will be difficult, but (a) you should work together, and (b) you will have the opportunity to rewrite. Homework given out Monday, returned Friday.

  On working together: Talking things out is a really good way to get over conceptual snags; usually it helps the explainer as much as, if not more than, the explainee; you come to understand something through explaining it, or sometimes you realize you DON'T understand something when you try to explain it... however, group work should be done after you've tried the problems on your own– don't just go and divvy up problems. And, of course, write-ups must be done individually.
- We'll use Strang's book, which I think has a nice conversational style. I also like Otto Bretscher's book *Linear Algebra* very much, and recommend consulting it if you like. You will get a lot out of reading the sections of the book before lecture–highly recommended, since you're going to read them anyway. Note: I don't love to lecture and will spend some in class doing group work. Thus, I may not get to everything. *You are responsible for what's in the assigned sections.*
- Please do interrupt me. I will inevitably use some piece of language that seems clear to me but is not. If you are confused, so is someone else.
- Solicit questions about the syllabus and the above.
- Most important: remember that this class is **supposed to be hard.** Some of the greatest minds of their times tore their hair out figuring this theory out. So it's not supposed to come to you easily. But it will come to you. Don't get discouraged.

- Pass out index cards, asking: name, phone, e-mail, hometown, major, and draw a picture of yourself with any identifying marks.

(15-20 min.)

2 Sections 1.1 and 1.2: Linear equations

I. Geometry of space

For the first couple of days, we will talk about linear algebra insofar as it involves the geometry of space. We'll start with three-dimensional space, because that's where (we think) we live. Once the mathematical tools are established, we'll freely use these ideas to talk about the geometry of higher-dimensional spaces, where (some think) we *actually* live!

In linear algebra, we will become accustomed to meeting systems of equations like:

$$\begin{aligned} x + y + z &= 8 \\ 3x + 2z &= 11 \\ x + 2y - 8z &= -11 \end{aligned}$$

We would like to solve this system, i.e. find values of (x, y, z) satisfying these three equations. Draw the picture of three-space; recall that a triple of numbers (x, y, z) corresponds to a point in this space, and the set of solutions to one of these equations corresponds to a plane. So what does the set of solutions to TWO of the equations look like? Three? (Draw picture of three intersecting planes.) So it seems very plausible that there is a unique solution to these equations.

Now what we'd like is something like

$$\begin{aligned} x \quad\quad\quad &= \\ y \quad\quad &= \\ z \quad &= \end{aligned}$$

That would be a system of linear equations we could very easily solve.

So we'd like only x in the first row, only y in the second row, only z in the third row. And we can get rid of the stuff we don't want by the elimination method. To wit: from $x + y + z = 8$ and $x + 2y - 8z = -11$ we subtract and get $-y + 9z = 19$. (Write THE WHOLE SYSTEM on the board again each time.=.) And subsequently subtract $3\times$ row one from row two to get $-3y - z = -13$. Then subtract 1/3 of second row from third row to get $(28/3)z = (70/3)$. Now we're in good shape? Divide the last row by 28/3 to get $z = 5/2$. Then add that to the second row and get $-3y = -21/2$. So divide by 3 and get $y = 7/2$. (Say the word "pivot" at this point.) Finally subtract second row from first, then third row, to get $x = 2$.

So as expected, the solution was one point! But note that this procedure was rather ad hoc and not a little tedious.

(30 min.)

II. Vectors.

We use *vector* notation, which you may or may not be familiar with. First of all, recall that we can write a point $\vec{v}$ in space as a triple of numbers–say, $(8, 11, -11)$. We will mostly think of this as a point; when it suits us, i.e. when there's addition, we'll draw the little arrow.

It will be convenient to write such a triple *vertically* as

$$\begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix} = \vec{v}.$$

Now we can say

$$5 \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix} = \begin{bmatrix} 40 \\ 55 \\ -55 \end{bmatrix}$$

is another name for "vector in the same direction at $\vec{v}$, but 5 times as far from the origin."

Of course, there's nothing special about the 5 here. The other important thing we can do is *add* vectors: for example,

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}.$$

Draw what this looks like geometrically; put the tail of one vector on the head of the next.

(35 min.)

2 minute contemplation. Consider the set of *all* vectors of the form

$$x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

where $x \in \mathbb{R}$ is some real number. (Note introduction of notation $\mathbb{R}$!) What are some examples? What, geometrically, does this set look like? How would things change if I replaced the vector above with some other vector?

Solicit responses, discuss.

2 minute contemplation. Consider the set of all vectors of the form

$$x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

with $x, y \in \mathbb{R}$. What are some examples? What, geometrically, does this set look like? How would things change if I replaced the pair of vectors above with some other pair of vectors?

[Note: this is as far as we got on the first day in spring 2000]

Solicit, discuss. Get to the point where we say that the two vectors produce a plane iff they don't lie on a line.

We write the original equation as

$$x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} = \vec{v}.$$

Now according to our dicussion above, we rather expect that the set of all

$$x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

forms a three-dimensional space–in other words, the whole world! So we expect once again that there *is* some choice of x, y, z making the equation hold.

(Although, from this point of view, it's not quite as clear why there's only one. Some people's geometric intuition will tell them this is so.)

Note also that this point of view–like the other geometric point of view–gives us no clue how to find x, y, z.

(50 min.)

III. Geometry of planes.

Just to give an idea; what can go wrong with the process? Do three planes necessarily meet in a point? Give an example where they don't: say $x - 2y + z = 4$ and $x - 2y + z = 5$ and $3x + 4y + 5z = 0$. Start to do the elimination–get the nonsensical $0 = 1$. If there's time, ask about the different ways the planes can intersect–but just walk around and discuss. If not, then end here.

LECTURE 2

0. Contemplation Ask about the question from last time, about the space spanned by two vectors.

I. Gaussian elimination This is no more than a systematization of the technique we used last time. So no new concepts here. The idea is: "work one column at a time, from left to right, transforming the augmented matrix we have into an augmented matrix of the sort we like." Another useful feature is that we never have to do anything but i) swap two rows or ii) add some multiple of a row to a **lower** row or iii) multiply a row by a **scalar** (define) The problem is, we may not be able to get it in such a nice form as the above. In fact, one notes that we can get it in that nice form exactly when our solution sets intersect in exactly one point!

Do the following example.

$$\begin{aligned} 3x + 6y - 3z &= 3 \\ x + 2y + 2z &= 3 \\ 2x + 4y - 5z &= 0 \end{aligned}$$

To eliminate, we will have to subtract 1/3 times the first column from the second column, and 2/3 times the first column from the third column. The number 3 here is called the **pivot**–it's the diagonal coefficient, by which we have to divide.

Now after eliminating, the first column, we are left with

$$\begin{aligned} 3x + 6y - 3z &= 3 \\ 2z &= 2 \\ -3z &= -2 \end{aligned}$$

Now there's a problem here, which is that our next pivot is zero! And we can't divide by zero! Hmm!

ASK: How many solutions are there? what sort of configuration of planes does this system represent? How can we tell?

If there's plenty of time, in fact, it might be good to break them into groups and ask them to figure out a way to describe all solutions to this equation. Maybe better to have them turn-to-the-neighbor and take 3 minutes to talk about it. I need them to start meeting each other at this point.

II. Matrices. Write up the eq'n from last time,

$$\begin{aligned} x + y + z &= 8 \\ 3x + 2z &= 11 \\ x + 2y - 8z &= -11 \end{aligned}$$

and point out that we might as well just save our time by not writing the x, y, and z, and writing instead

$$\begin{array}{cccc} 1 & 1 & 1 & 8 \\ 3 & 0 & 2 & 11 \\ 1 & 2 & -8 & -11 \end{array}$$

with the understanding that we can now add and subtract multiples of rows from one another, and we are aiming to wind up with something like

$$\begin{array}{cccc} 1 & & & * \\ & 1 & & * \\ & & 1 & * \end{array}$$

Note that if we have

$$\begin{array}{cccc} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \end{array}$$

we're in business. So we see that what we're doing is really manipulating rectangular arrays of numbers in certain ways, and that motivates the following definition.

Definition: A *matrix* is an array of numbers. If a matrix A has m rows and n columns, we say A is an $m \times n$ matrix.

It is customary to refer to the number in row i and column j as a_{ij}.

Examples:

- $A = \begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix}$ (2×2)

- $A = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix}$ (3×1)
- $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & -8 \end{bmatrix}$ (3×3)

Now let me remind you that the equation above could be rewritten as

$$x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}.$$

That motivates the following definition.
Let A be an $m \times n$ matrix, and $\vec{x}$ a length n vector. Write

$$A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix}$$

and

$$\vec{x} = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$

So comment here: each $\vec{v}_i$ is a length m vector, which is to say a $m \times 1$ matrix. Each x_i is a *number*.

Then

Definition: The product $A\vec{x}$ is defined to be the sum

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \sum_{i=1}^{n} x_i\vec{v}_i.$$

(Remark that they should get used to sigma notation, if they're not.)

Emphasize that this is just a new piece of notation. But a useful one–for example,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}.$$

So we can now write our equation in an even *simpler* form: we are trying to find a vector $\vec{x} = \vec{x}yz$ such that

$$A\vec{x} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}$$

(where A is the 3×3 matrix above.)

Note that I've spent the last two classes giving you more and more ways to phrase this equation, without giving you any more insight into how to solve it!

(But **remark** at this point, and write on the board: one could solve the equation if one knew how to "divide by A"! Hold that thought...)

Another important example: the dot product of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is nothing more than the matrix product

$$[x_1 x_2 x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1[x_1] + y_2[x_2] + y_3[x_3] = [x_1y_1 + x_2y_2 + x_3y_3].$$

(Note: on 2 Feb 2000, this was as far as I got.)

III. Linear transformations

I'm going to jump ahead a little bit because it makes what follows make much more sense.

First of all, notation: $\mathbb{R}^n$ is the set of all length n vectors. (They've probably seen this before.)

Now suppose we have a matrix A. Then, given any length n vector $\vec{x} \in \mathbb{R}^n$, we can produce a length m matrix $A\vec{x}$.

In other words, *an $m \times n$ matrix yields a rule for turning length n vectors into length m vectors.* Or, if you like, the matrix defines a function $T : \mathbb{R}^n \to \mathbb{R}^m$.

So we TRANSLATE the question of "is there an $\vec{x}$ such that $A\vec{x} = \vec{b}$?" to "is $\vec{b}$ in the image of T"?

Some good examples: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. To figure out what functions these matrices describe is a great groupwork. I'll do one of them in class today, and then do the rest for groupwork at the beginning of next time.

Next time: matrix multiplication.

LECTURE 3

This is going to be a groupwork day. First, I talk about the notion of a matrix yielding a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, which I didn't talk about yesterday.

Ex: The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

multiply it by a vector $\vec{x}$ and see what you get. You get $\vec{x}$ again. This is called the *identity matrix*–it leaves everything the same.

GROUPWORK: Combining matrices

Let $\vec{x}$ be the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Let A be the $3x3$ matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and B be the $3x3$ matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & -8 \end{bmatrix}.$$

- What is $A\vec{x}$?
- What is $B\vec{x}$?
- What is $A(B\vec{x})$? (That is: what happens if we multiply $\vec{x}$ by the matrix B, and *then* by the matrix A?
- Is there some single matrix C that does the combined job of A and B; that is, such that $C\vec{x} = A(B\vec{x})$?
- If you found such a C, is it the only one? Or are there several choices of C such that $C\vec{x} = A(B\vec{x})$?

So the fact is: given *any* two matrices A and B (of any size!), there is a matrix such that $C\vec{x} = A(B\vec{x})$.

Definition. Let A be an $m \times n$ matrix and B a $n \times p$ matrix. Then AB is the $m \times p$ matrix such that $AB\vec{x} = A(B\vec{x})$ for all $\vec{x} \in \mathbb{R}^p$.

Note that we have *not* yet proved such a matrix exists! That will wait until we discuss linear transformations in earnest. For now, believe that it does.

Computing the matrix product.

Two good ways:

$$A \begin{bmatrix} | & | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\vec{v}_1 & \dots & A\vec{v}_n \\ | & | & | \end{bmatrix}$$

Or, my favorite way:

$$(\begin{bmatrix} - & \vec{w}_1 & - \\ - & \vdots & - \\ - & \vec{w}_m & \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix})_{ij} = \vec{w}_i \cdot \vec{w}_j$$

Do the example from the groupwork in each of these ways. If there's time, remark on the non-commutativity of matrix multiplication. There's not time to say too much–only that it's to be paid careful attention to, and that it's the reason for things like the uncertainty principle.