

Second mid-term exam for Math 204

June 22, 2000

Rules and regulations

- This exam is due at noon on Wednesday, April 12.
- The exam is open-book; we ask you not to consult books other than your textbook, or other people. We also ask that you do not use computing devices for the problems on the exam.
- If you feel there's a typo on the exam, or that a question is unclear, please e-mail your section leader right away—your promptness will benefit your fellow students.
- Please take the time to write clearly and in complete sentences, especially when you are writing a proof. And don't forget to check your arithmetic—you have plenty of time to make sure everything is right.
- Remember to write and sign the honor pledge on the front of your exam.
- Good luck and have a good time!

Problems

1. ROTATIONS AND AXES

We define a *rotation* to be an orthogonal matrix which has determinant 1.

a. Give an example of a 3×3 permutation matrix, other than the identity, which is a rotation. What are the eigenvalues of this matrix? What are the eigenvectors?

b. Give an example of a 3×3 rotation A such that

$$A\vec{e}_1 = \vec{e}_1,$$

where \vec{e}_1 is the standard basis element $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. What are the eigenvalues of A ? What are the eigenvectors?

c. Give an example of a rotation of the form

$$A = \begin{bmatrix} 2/7 & a & b \\ 3/7 & c & d \\ 6/7 & e & f \end{bmatrix}.$$

d. Here is a false statement: if A is a rotation, the eigenvalues of A are all ± 1 . Here is a fake proof of the false statement. Suppose \vec{v} is an eigenvector for A , with eigenvalue λ . Then

$$A\vec{v} = \lambda\vec{v}.$$

By Strang's theorem 3R (p.168), and the fact that A is orthogonal, we have

$$\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = \lambda^2\|\vec{v}\|.$$

Since \vec{v} is an eigenvector, it is nonzero, so $\|\vec{v}\|$ is nonzero; dividing out, we get

$$\lambda^2 = 1$$

which yields $\lambda = \pm 1$.

Identify the incorrect step in the fake proof, and explain why it is incorrect.

Physically speaking, an axis of a rotation is a line which is left unchanged by the rotation. (For instance, the axis of the rotation of the Earth on its axis is the line joining the North and South Poles.) We express this idea mathematically by defining an *axis* of a rotation A to be a nonzero vector \vec{v} such that $A\vec{v} = \vec{v}$. In other words, an axis of A is an eigenvector for A with eigenvalue 1.

e. Prove that every 3×3 rotation has an axis. (If your answers to parts a. and b. do not support this conclusion, check your work!)

2. TRUE AND FALSE

Below you will find a series of assertions, some true and some false. In each case, you should tell us whether the statement is true or false and provide a justification for your answer. Note that all matrices in this problem are understood to be *square*.

a. There exists a matrix A such that $A^{17} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

b. If P is a projection, and \vec{v} a vector, then

$$\|P\vec{v}\| \leq \|\vec{v}\|.$$

c. If A is a matrix and U its row-reduced form, then A and U have the same determinant.

d. If A is a matrix and U its row-reduced form, then A and U have the same eigenvalues.

e. If A is an $n \times n$ matrix and I is the $n \times n$ identity matrix, then A and $A + I$ have the same eigenvectors.

- f. If A is a diagonalizable matrix, its eigenvalues are distinct.
g. There exist subspaces V and W of \mathbb{R}^4 such that $\dim V = 2$, $\dim W = 2$, and $\dim(V \cap W) = 1$.

3. COMPUTING A DETERMINANT.

Consider an $(n \times n)$ matrix

$$\Omega_n = \begin{pmatrix} 1 & -\frac{\omega_1}{\omega_1+\omega_2} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{-\omega_3}{\omega_2+\omega_3} & 1 & \frac{-\omega_2}{\omega_2+\omega_3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & \ddots & & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & \frac{-\omega_{i+1}}{\omega_i+\omega_{i+1}} & 1 & \frac{-\omega_i}{\omega_i+\omega_{i+1}} & 0 & \cdots \\ \vdots & & & \ddots & & & & & \ddots & \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{-\omega_{n+1}}{\omega_n+\omega_{n+1}} & 1 \end{pmatrix}.$$

where $\omega_1, \dots, \omega_{n+1}$ are arbitrary real numbers.

For example,

$$\Omega_3 = \begin{bmatrix} 1 & -\frac{\omega_1}{\omega_1+\omega_2} & 0 \\ -\frac{\omega_3}{\omega_2+\omega_3} & 1 & -\frac{\omega_2}{\omega_2+\omega_3} \\ 0 & -\frac{\omega_4}{\omega_3+\omega_4} & 1 \end{bmatrix}.$$

Prove by induction that

$$\det \Omega_n = \frac{\omega_2 \omega_3 \cdots \omega_n (\omega_1 + \omega_2 + \cdots + \omega_{n+1})}{(\omega_1 + \omega_2)(\omega_2 + \omega_3) \cdots (\omega_n + \omega_{n+1})}.$$

(Comment: if you're having trouble, try proving the desired fact for Ω_2 and Ω_3 .)

4. HEALTH MINISTER

You are the health minister of a developing country. A new infectious disease has broken out, whose spread you have modeled in the following way. Your population is divided into three groups: rural, transient (truck drivers, merchants) and urban. The greatest likelihood of contagion occurs among the closely-packed urban population; on the other hand, the lowest rate of cure occurs among the hard-to-treat rural population. Meanwhile, the transient group is at a high risk of infection from both the urban and rural inhabitants with whom they come in contact on their travels. Quantitatively, we estimate that in a given month, on average:

- Each infected rural inhabitant infects 1 other rural inhabitant and 1 transient;
- Each infected transient infects 1 rural inhabitant, 1.5 other transients, and 2 urban inhabitants;
- Each urban inhabitant infects 9 other urban inhabitants and 2 transients.

Fortunately, there is a treatment for this disease—but it is easier to obtain, and more effective, in the cities. In particular, each month on average

- 50% of the infected rural inhabitants are cured or recover;
- 60% of the infected transients are cured or recover;
- 90% of the infected urban inhabitants are cured or recover.

For computational purposes, assume that in each month the cures and recoveries take place first, and the new infections second. For example: suppose that at the beginning of the month there are 1000 infected urbanites and no infected rural inhabitants or transients.

The first thing that happens is that 90% of the infected urban inhabitants are cured or recover, leaving 100 infected. Now each of those 100 infects 9 urban inhabitants and 2 transients; so at the end of the month there are 0 infected rural inhabitants, 200 infected transients, and 1000 infected urbanites.

What happens the next month? Now 90% of the urbanites and 60% of the transients are cured or recover, leaving 100 infected urbanites and 80 infected transients. The 100 urbanites infect 900 more urbanites and 200 transients. The 80 transients infect 80 rural inhabitants, 120 more transients, and 160 urban inhabitants. In total, there are now 80 infected rural inhabitants, $80 + 200 + 120 = 400$ infected transients, and $100 + 900 + 160 = 1160$ infected urbanites.

a. Find a 3×3 matrix A describing the month-by-month evolution of this disease. More precisely, if

$$\vec{v}_n = \begin{bmatrix} \text{infected rural inhabitants at month } n \\ \text{infected transients at month } n \\ \text{infected urban inhabitants at month } n \end{bmatrix}$$

then A should satisfy

$$\vec{v}_{n+1} = A\vec{v}_n.$$

Note that A is *not* a Markov matrix.

NOTE: You really want your answer to this question to be correct—otherwise, you will not be able correctly to carry out the analysis in the rest of the problem. You should check (you need not include these computations in your paper) that

$$A \begin{bmatrix} 0 \\ 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \\ 1000 \end{bmatrix}$$

and

$$A \begin{bmatrix} 0 \\ 200 \\ 1000 \end{bmatrix} = \begin{bmatrix} 80 \\ 400 \\ 1160 \end{bmatrix}.$$

b. What are the eigenvalues and eigenvectors of A ?

c. Is the disease dying out or becoming an epidemic? Suppose we have 1000 infected urbanites at the start of month 0 and no one else infected. Approximately how many people will be infected at the start of month 10? How many of these will be rural, how many transient, and how many urban?

d. Your two deputy health ministers, Minister McCulloch and Minister Malkiel, come to you with two different plans. Minister Malkiel argues that the serious problem lies in the “disease reservoir” of the hard-to-treat rural population and in the transient population, which keeps the disease moving between different groups. He proposes an expensive plan which would cause the cure-and-recovery rate in the rural areas to increase from 50% to approximately 100%, and the rate among the transients from 60% to 80%.

Write down a matrix B that would describe the month-by-month evolution of this disease if the cure-and-recovery rates were improved according to Minister Malkiel’s plan. What are the eigenvalues of B ? Would Malkiel’s plan end the epidemic?

e. Minister McCulloch proposes an equally expensive plan to increase the potency of the medicine used. This affects every population roughly equally; the cure-and-recovery rates would go up to 75% in the rural region, 80% in the transient population, and 95% among the urban population.

Write down a matrix C that describes the month-by-month evolution of the disease under McCulloch’s plan. What are the eigenvalues of C ? Would McCulloch’s plan end the epidemic?

f. (Extra credit) Now return to our original scenario, described by the matrix A . Prove that the disease will become an epidemic under *any* choice of initial conditions \vec{v}_0 .