

Final exam for Math 204, spring 2000

10 May 2000

This is the final examination for Math 204. You have three hours to complete the test. One of your instructors will be present for the first half-hour in order to answer any questions that may arise. Please look over the whole exam before you start working. The questions will be weighted approximately equally, but some are more difficult than others—we recommend working the easier problems first. Good luck!

1a. Let A be the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Give a basis for the column space of A and the nullspace of A . What is the rank of A ?
What is the dimension of the row space of A ?

1b. Let B be the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 4 \end{bmatrix}$$

Give a basis for the nullspace of B . What is the rank of B ?

2. Let P_3 be the vector space of polynomials of degree less than or equal to 3, and let P_4 be the vector space of polynomials of degree less than or equal to 4. That is, P_3 is the space of all polynomials of the form

$$a_0 + a_1t + a_2t^2 + a_3t^3,$$

and P_4 is the space of all polynomials of the form

$$a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4.$$

a. What are the dimensions of P_3 and P_4 ?

b. Let B_3 be the basis $\{1, t, t^2, t^3\}$ for P_3 , and let B_4 be the basis $\{1, t, t^2, t^3, t^4\}$ for P_4 . Let T be the linear transformation from P_3 to P_4 defined by

$$T(f) = (1 + t)f.$$

What matrix represents the transformation T with respect to B_3 and B_4 ?

c. Let B'_3 be the basis $\{1, (1 + t), (1 + t)^2, (1 + t)^3\}$ for P_3 , and let B'_4 be the basis $\{1, (1 + t), (1 + t)^2, (1 + t)^3, (1 + t)^4\}$ for P_4 . What matrix represents the transformation T with respect to B'_3 and B'_4 ?

3. Find a least-squares solution \vec{x} to $A\vec{x} = \vec{b}$, and compute $\vec{p} = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $\vec{b} - \vec{p}$ is perpendicular to the columns of A .

4. Indicate whether each of the following statements is true or false. *You need not justify your answers.*

- T** **F** If A is a real square matrix, A^T and A have the same eigenvalues.
- T** **F** If A is a square matrix, A^H and A have the same eigenvalues.
- T** **F** If A is a positive definite real symmetric matrix, then the trace of A is a positive real number.
- T** **F** If A is a square matrix whose nullspace consists only of the zero vector, A is invertible.
- T** **F** If A is the matrix $\begin{bmatrix} 1/10 & 2/10 \\ 3/10 & 4/10 \end{bmatrix}$, then $\lim_{n \rightarrow \infty} A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- T** **F** If A is a real square matrix whose eigenvalues are all equal to ± 1 , then A is orthogonal.
- T** **F** The matrix $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ can be expressed as $A^T A$ for some real 2×2 matrix A .
- T** **F** Every real orthogonal matrix is also unitary.
- T** **F** There exists a matrix A such that $A + cI$ is invertible for all complex numbers c .

5. Rabbits and wolves. Let $r(t)$ be the population of rabbits at time t , and $w(t)$ the population of wolves. Wolves like to eat rabbits; thus, the population of rabbits is diminished proportionally to the number of wolves, and the number of wolves increases proportionally to the number of rabbits. All in all, we propose that the number of rabbits and wolves are governed by the following differential equation:

$$dr/dt = 4r - 2w, dw/dt = r + w.$$

- a. Express this differential equation in matrix form $d\vec{y}/dt = B\vec{y}$.
- b. Are the rabbit and wolf populations dying out, burgeoning out of control, or neither?
- c. If the initial conditions are $r(0) = 300, w(0) = 200$, what are the populations at time t ?
- d. After a long time, what is the proportion of rabbits to wolves?
- e. Your gardener comes up with a plan to control the rabbit-wolf population; he will drug the rabbits, making them easier for the wolves to catch and eat. This should have the effect of replacing the original differential equation with

$$dr/dt = 4r - (13/2)w, dw/dt = r + w.$$

Describe the long-term behavior of the populations predicted by this differential equation. How do you know that this prediction is unrealistic?

6. Let $q(x, y) = 13x^2 + 24xy + 13y^2$.

a. Find a matrix A such that

$$q(x, y) = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix}.$$

b. The solution set of $q(x, y) = 1$ is an ellipse in the $x - y$ plane. Sketch this ellipse, and compute the length of its axes.

c. Suppose we modify the quadratic form, changing it to

$$q(x, y) = cx^2 + 24xy + 13y^2$$

for some number c . For which value or values of c is it the case that q is positive semidefinite, but not positive definite?

7. Define matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Determine whether each of the properties below applies to A , B , both, or neither. Justify your answers.

The properties: orthogonal, invertible, permutation, Hermitian, diagonalizable, in Jordan normal form, rank 1.

8. a. Let A be the matrix $\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A and A^T .

b. Let M be a *Markov matrix*; that is, M is a square matrix, all of whose entries are real non-negative numbers, and each of whose columns sum to 1. An example of a Markov matrix is the matrix A from part **a**. Prove that 1 is an eigenvalue for every Markov matrix M . (Hint—it is enough to prove that 1 is an eigenvalue for M^T .)

c. Prove the following statement: if M is a Markov matrix, and λ is an eigenvalue of M , then $|\lambda| \leq 1$.

Remark: This is a difficult question, and if you're having trouble proving the general theorem in **c** you may want to try a special case. For instance—can you prove the statement if you assume that M is 2×2 ? Can you prove the statement assuming that λ is a *real* eigenvalue? Can you prove the statement assuming that M is symmetric?

Remark 2: Strang asserts statements **b** and **c** without proof in your textbook—be advised that “Strang says the statement is true” is not sufficient justification for the purposes of this exam.

Remark 3: Don't be concerned if you're not familiar with Strang's discussion of Markov processes—you will not need anything from that section to prove statements **b** and **c**.