Section 11.4: Radius of convergence

Today: a 20 minute groupwork. Last week was more theory, this week more practice, and so we will do more groupwork this week.

A sum-up of what we did last week.

I. Approximated a function f by a Taylor polynomial $p(x)$ of degree n. II. Used Lagrange's Theorem to show that as the number of terms of $p(x)$ grows larger and larger, the approximation to $f(x)$ gets closer and closer (sometimes.)

Recall again the example of $1/(1-2x)$, and how it converged for $|2x| < 1$. Draw the real number and draw a picture of the *radius of convergence.*

Ex: Consider the infinite series

$$1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \dots$$

For what values of x does this converge? Ask around, how do we test if such a series converges? The factorials should suggest that the (absolute) ratio test is in order. We have $a_n = (-1)^n x^{2n}/(2n)!$ and so

$$|a_{n+1}/a_n| = |\frac{(-1)^{n+1}x^{2n+2}(2n)!}{(-1)^n x^{2n}(2n+2)!}| = x^{2n}/2n(2n+1).$$

Now what happens to this as n gets large? Take a vote. Split in pairs.

Yes–this goes to 0 whatever x is, so the series converges for all x. Now describe an alternate way to do this, using absolute value test and then regular ratio test.

EXERCISE: The series above converges to $\cos x$ for all x. (Use Lagrange's theorem) (In Fall 98, they did this on homework.)

Groupwork:

1. Compute the Taylor series for $(1+x)^{1/2}$ near $x = 0$.

2. For which x does this series converge?

Now notice that in each case the radius of convergence is of the form $|x| < R$. In other words, it's just a symmetric interval around $x = 0$. Draw something the *can't* arise; a domain with holes.

This is expressed in the following **Theorem.** Suppose

$$a_0 + a_1 c + a_2 c^2 + \dots$$

converges. Then

$$a_0 + a_1 x + a_2 x^2 + \dots$$

converges for all x with $|x| < |c|$.

Proof. Use the absolute limit comparison test. Let $u_n = a_n c^n$, and $v_n = a_n x^n$. We know that

$$\sum_0^\infty u_n$$

converges. Now

$$\lim_{n\to\infty} |u_n/v_n| = \lim_{n\to\infty} |x^n/c^n| = \lim_{n\to\infty} |x/c|^n$$

Since $|x/c| < 1$, this limit is 0, and we are done.

Section 11.5: Manipulating power series

Definition: A *power series* in x is an infinite series of the form

$$a_0 + a_1 x + a_2 x^2 + \dots$$

(i.e., an "infinite polynomial.")

Theorem: Suppose

$$a_0 + a_1 x + a_2 x^2 + \dots$$

converges to a function $f(x)$. (For instance, we might know this because the series is geometric, or by Lagrange.) Then

1. $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$
2. $\int_0^x f(t)dt = a_0 x + (1/2)a_1 x^2 + (1/3)a_2 x^3 + \dots$

We won't prove this theorem in class. But here's a question: why isn't this obvious? We *know* that if we differentiate a sum, we can differentiate each term. But careful: *none of what we know about addition* automatically applies to infinite series! As we saw two weeks ago, where we couldn't necessarily re-order the terms of an infinite sum. We'll *not* prove this theorem here.

Ex: We can find that

$$1/(1+x) = 1 - x + x^2 - x^3 + \dots, |x| < 1$$

(Ask if this is all right, though. If there's confusion, get a student to explain it.) Then plugging in x^2, we get

$$1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \dots, |x| < 1$$

And finally, integrating, we get

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \dots, |x| < 1$$

ASK: Does the RHS converge when $x = 1$?

IN FACT, though we will not prove it, the above is true even when $x = 1$. This yields the famous identity (prompt for LHS)

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$$

Groupwork:

Using the Taylor series for $1/(1+x)$ above,

1. compute a Taylor series for $log(1+x)$ at $x = 0$. What do you expect is the value of

$$1 - 1/2 + 1/3 - 1/4 + \dots ?$$

2. compute a Taylor series for $1/(1+x)^2$ at $x = 0$. What is the value of

$$\sum_0^\infty n/5^n = 0 + 1/5 + 2/25 + 3/125 + 4/625 + \dots ?$$

Limits and power series

Power series are often a convenient way to compute limits. Example. We know that for all x,

$$\sin x = x - x^3/3! + x^5/5! - \dots$$

Therefore, it is also the case that

$$\sin x/x = 1 - x^2/3! + x^4/5! - \dots$$

for all x *except 0.* Because it's just not kosher to divide both sides by 0. However, we can see hereby that

$$\lim_{x\to 0} \sin x/x = \lim_{x\to 0} 1 - x^2/3! + x^4/5! - \dots = 1.$$

Of course, we already knew this by L'Hospital's rule. But if you know the Taylor series, it is easier to write this down than to take derivates. All the more so if L'Hospital's rule would require multiple differentiations.

0.1 Algebra of power series

Theorem (Stated loosely) Suppose

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \dots, |x| < R \\ g(x) &= b_0 + b_1x + a_2x^2 + \dots, |x| < R \end{aligned}$$

Then $f(x) + g(x), f(x) - g(x), f(x)g(x), f(x)/g(x)$ "are what you think"; e.g.

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots, |x| < R.$$

Now I want to talk more carefully about multiplication and division.

For instance, suppose we wanted a power series converging to

$$(1 + x^2)e^x.$$

We write

$$\begin{aligned} (1+x^2)e^x &= (1+x^2)(1+x+x^2/2+x^3/6+x^4/24\dots) \\ &= (1+x+x^2/2+x^3/6+x^4/24+\dots)+(x^2+x^3+x^4/2+\dots) \\ &= (1+x+3x^2/2+7x^3/6+13x^4/24+\dots), |x| < 1 \end{aligned}$$

It's not so easy to see the pattern there, but anyway these are the first four terms.

What about division? Try

$$e^x/(1 + x^2).$$

We could do

$$e^x/(1+x^2) = (1 - x^2 + x^4 - x^6 + \dots)(1 + x + x^2/2 + \dots).$$

And that would work. We could also reason: Suppose

$$e^x/(1+x^2) = a_0 + a_1x + a_2x^2 + \dots, |x| < 1.$$

Then

$$(1+x^2)(a_0 + a_1x + a_2x^2 + \dots) = e^x = (1 + x + x^2/2 + \dots)$$

Or

$$(a_0 + a_1x + a_2x^2 + \dots) + (a_0x^2 + \dots) = (1 + x + x^2/2 + \dots)$$

So we get

$$a_0 = 1, a_1 = 1, a_2 + a_0 = 1/2$$

and conclude $a_2 = -1/2$.

Groupwork: Find the first five terms of a power series converging to $1/(1-x-x^2)$ using the method above: find a power series

$$a_0 + a_1 x + a_2 x^2 + \dots$$

such that

$$(a_0 + a_1 x + a_2 x^2 + \dots)(1 - x - x^2) = 1.$$