

Section 11.1: Taylor series

Today we're going to begin the development of the remarkable theory of Taylor series. We'll use the development of infinite series we've already done, and combine this with our understanding of calculus so far.

(Remark: more on infinite series with positive and negative terms later on this week.)

Main idea: Make all functions as easy to understand as polynomials.

We start by *approximating* a given function by polynomials.

PROBLEM: Approximate the function $f(x) = e^{-x}$ near $x = 0$ by a linear function $\ell(x) = a_0 + a_1x$. (Draw a picture.)

What do we mean by "approximate"? Well, it seems reasonable to suppose

$$\ell(0) = f(0).$$

In other words,

$$a_0 = e^{-0} = 1.$$

Moreover, we might want to set

$$\ell'(0) = f'(0)$$

or

$$a_1 = -e^{-x}|_{x=0} = -1.$$

We conclude that

$$\ell(x) = 1 - x$$

is a good approximation near $x = 0$.

OK, what if we wanted to approximate f by a *quadratic* function $q(x) = a_0 + a_1x + a_2x^2$?

Then we can once again set $q(0) = f(0)$ and $q'(0) = f'(0)$. And once again we get $a_0 = 1$ and

$$q'(0) = (a_1 + a_2x)|_{x=0} = a_1 = f'(0) = -1.$$

Now we still get to choose what a_2 is, so let's stipulate further that

$$q''(0) = f''(0)$$

or

$$2a_2 = e^0 = 1$$

so $a_2 = 1/2$, and we end up with

$$q(x) = 1 - x + x^2/2.$$

In general:

To approximate a function f near $x = a$ by a polynomial $p(x)$ of degree d , find a choice of $p(x)$ such that

$$\begin{aligned} p(0) &= f(0) \\ p'(0) &= f'(0) \\ p''(0) &= f''(0) \end{aligned}$$

and so on unto

$$p^{(d)}(0) = f^{(d)}(0)$$

(remark this notation.)

This should take 15 minutes

Groupwork:

1. Approximate $f(x) = e^{-x}$ near $x = 0$ by a polynomial of degree 3.
2. Approximate $f(x) = 1/(1 - 2x)$ near $x = 0$ by a polynomial of degree 2.
3. Approximate $f(x) = 1/(1 - 2x)$ near $x = -1$ by a polynomial of degree 2.

(Note: answer to third question is $19/27 + (14/27) * x + (4/27) * x^2$.)

If this isn't enough for some groups, ask them to try to find a pattern for the first one. Give 23 minutes. Meanwhile, find spokesperson. Then give 12 minutes to present solutions.

Section 11.2: The error in Taylor series

Return to the function $f(x) = 1/(1 - 2x)$. Last time we found that a good quadratic approximation for $f(x)$ was

$$1 + 2x + 4x^2.$$

Suppose we carried this argument further, and tried to compute a polynomial approximation (Taylor polynomial) of degree d . So we want

$$p^{(n)}(0) = f^{(n)}(0)$$

for all $n < d$. If

$$p(x) = a_0 + a_1x + \dots + a_dx^d$$

then

$$\begin{aligned} p'(x) &= a_1 + 2a_2x + 3a_3x^2 + \dots + da_dx^{d-1} \\ p''(x) &= 2a_2 + 6a_3x + \dots + d(d-1)a_dx^{d-2} \end{aligned}$$

and so on. And in general

$$p^{(n)}(x) = (n!)a_n + \dots$$

So we get

$$(n!)a_n = f^{(n)}(0)$$

for *all* functions f , when approximating near $x = 0$. In this case, what is $f^{(n)}$? Well,

$$\begin{aligned} f'(x) &= (-2) * (-1)(1 - 2x)^{-2}, \\ f''(x) &= (-2) * (-1) * (-2) * (-2) * (1 - 2x)^{-3}, \\ f^{(3)}(x) &= (-2) * (-1) * (-2) * (-2) * (-2) * (-3) * (1 - 2x)^{-4} \end{aligned}$$

and in general

$$f^{(n)}(x) = (2^n)n!(1 - 2x)^{-n-1}$$

so that $f^{(n)}(0) = 2^n n!$. Conclude that $a_n = 2^n$, so

$$1 + 2x + 4x^2 + 8x^3 + \dots + 2^d x^d$$

is the degree d approximation to $1/(1 - 2x)$.

IN BOOK: How to modify this approach to approximating a function near $x = a$. We end up with

$$f(a) + f'(a)(x - a) + f''(a)(x - a)^2/2 + \dots + f^{(d)}(a)(x - a)^d/d!.$$

Let me make a remark.

$$1 + 2x + 4x^2 + 8x^3 + \dots$$

is a *geometric series!* As long as $|2x| < 1$, in other words, this series converges to $1/(1-2x)$! In other words, if we keep on approximating by higher and higher degree polynomials, we get closer and closer to the actual function $f(x)$, in this case. Is this true in general?

Let $f(x)$ be a function with a Taylor polynomial at $x = a$

$$f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2 + \dots + f^{(n)}(a)(x-a)^n/n!$$

Let $R_n(x; a) = f(x) - p(x)$. (The *error function*).

Theorem. (Lagrange) There is a number c_n between a and x such that

$$R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}.$$

What on earth does this mean? Well, in general it is a bound; the point is that $f^{(n+1)}(c_n)$ hopefully does not get too big.

Ex: Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$ for all n . So the Taylor polynomial has $a_n = 1/n!$; that is

$$1 + x + x^2/2! + x^3/3! + \dots + x^n/n!$$

What does Lagrange tell us? Well, that

$$R_n(x; 0) = f(x) - p(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}x^{n+1}$$

Or, in other words

$$\frac{e^{c_n}}{(n+1)!}x^{n+1}$$

Now note that $e^{c_n} \leq e^x$, so we get

$$f(x) - p(x) \leq e^x x^{n+1}/(n+1)!$$

As n gets large, this approaches 0. In other words, the value of $p(x)$ approaches $f(x)$ as more and more terms are added. In other words, the infinite sum

$$1 + x + x^2/2! + x^3/3! + \dots$$

converges to e^x . (We already knew it converged for $x \geq 0$, by the ratio test, but we had no idea to what.)

What about when $x < 0$? This is a good time to remark on the *absolute value test*.

bf Theorem. Let $\sum_{n=1}^{\infty} a_n$ be a series such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Then $\sum_{n=1}^{\infty} a_n$ converges.

Note that this is very like the absolute value test for improper integrals, and the proof is quite similar as well.

So consider the series above when $x < 0$. We can now point out that the sum of the absolute values converges, whence we are done by the test above. (Write this out on board.)

We can also use this example to make very good approximations to e . For instance, taking

$$1 + 1 + 1/2! + 1/3! = 22/3 = 2.6666\dots$$

is off by at most

$$e^1/24 = e/24 < 1/8.$$

So we get

$$2.6666 < e < 2.791666.$$

QUESTION: Is it true that, for all x , and all infinitely differentiable f , that the Taylor series

$$f(a) + f'(a)(x - a) + f''(a)(x - a)^2/2 + \dots$$

converges to $f(x)$?

Take a vote. Coalesce in pairs, converse, bring back to front. We should see that this is not true for $1/(1 - 2x)$ when $x > 1/2$.

But it gets worse; comment that if $f(x) = e^{1/x^2}$, the Taylor series is 0. But that's the last we'll hear of such monsters in this class.

Lagrange's Theorem

Today we'll talk about the mysterious theorem above and how to prove it.

bf Plausibility argument. (Not a proof!) Suppose we believed that the Taylor series converged to the function, i.e. that

$$f(a) + f'(a)(x - a) + \dots + f^{(n)}(a)(x - a)^n/n! + f^{(n+1)}(a)(x - a)^{n+1}/(n + 1)! + \dots = f(x).$$

(We don't *know* this, of course—just bear with me!) Now mark the first n terms as $p(x)$, so that the rest is $R_n(x; a)$. The idea is as follows. We figure the terms are getting smaller and smaller really fast by the time we're out at n ; in particular, maybe we think the $R_n(x; a)$ part mostly consists of the first term!

So then we'd estimate

$$R_n(x; a) \sim f^{(n+1)}(a)(x - a)^{n+1}/(n + 1)!$$

which already looks a lot like Lagrange's estimate. So the idea is that to make this estimate exact, we replace a by something pretty close to a , namely c_n .

Now let's look at something like a proof.

Let $p(x)$ be a degree- n Taylor polynomial for $f(x)$ at $x = a$, and let $R_n(x; a) = p(x) - f(x)$.

Recall that $p^{(i)}(a) = f^{(i)}(a)$ for all $i \leq n$. So

$$R_n(x; a)^{(i)}(a) = p^{(i)}(a) - f^{(i)}(a) = 0$$

for all $i \leq n$. So look at this function; its first derivative, second derivative, etc. are all zero! So how big can it possibly get by the time we get to x ?

Lemma. Let $R(x)$ be a function whose first $n + 1$ derivatives exist on the interval $[a, b]$. Suppose

$$R^{(i)}(a) = 0$$

for all $i \leq n$. Let M be the maximum value of $R^{(n+1)}(x)$ on $[a, b]$, and m the minimum value.

Then

$$m(b - a)^{n+1}/(n + 1)! \leq R(b) \leq M(b - a)^{n+1}/(n + 1)!$$

Talk about what this says when $n = 0$; draw a function wiggling between two lines.

Following the book, we'll show this when $n = 2$. So now. We know

$$R(a) = R'(a) = R''(a) = 0, m \leq R'''(a) \leq M.$$

So we're going to work backwards. Notice that we already have bounds on the function R''' . And we want bounds on the function R . So let's step backwards through the derivatives. Note that by F T o' C,

$$R''(t) = \int_a^t R'''(x)dx$$

and

$$m(t-a) \leq \int_a^t R'''(x)dx \leq M(t-a).$$

So

$$m(x-a) \leq R''(x) \leq M(x-a).$$

Pretty good! Now do the same thing again.

$$R'(t) = \int_a^t R''(x)dx$$

and

$$\int_a^t m(x-a)dx \leq \int_a^t R''(x)dx \leq \int_a^t M(x-a)dx$$

or

$$m(x-a)^2/2|_a^t \leq R'(t) \leq M(x-a)^2/2|_a^t$$

or

$$m(t-a)^2/2 \leq R'(t) \leq M(t-a)^2/2.$$

So we've got bounds for $R'(t)$. Finally, do it one more time, and get

$$m(t-a)^3/6 \leq R(t) \leq M(t-a)^3/6.$$

the desired result.

Now we return to the Lagrange formula. Our error function $R_n(x; a)$ is exactly of the sort described in the Lemma. So let m be the minimum value of $f^{(n+1)}$ on $[x, a]$ and M the maximum value. Then by the Lemma,

$$m(x - a)^{n+1}/(n + 1)! \leq R_n(x; a) \leq M(x - a)^{n+1}/(n + 1)!$$

or, in other words,

$$m < R_n(x; a)(n + 1)!/(x - a)^{n+1} < M.$$

Now draw the graph of $f^{(n+1)}$ on $[a, x]$ and observe that, by the intermediate value theorem, if $m < q < M$, there is some c in $[a, x]$ such that $f(c) = q$. In other words, there is a c_n in $[a, x]$ such that

$$R_n(x; a)(n + 1)!/(x - a)^{n+1} = f^{(n+1)}(c_n)$$

which is the desired result.

If there's extra time, show e^{-1/x^2} . Don't forget to leave ten minutes to pass out the Questionnaire!