Section 10.4: The integral test

Let's return to the example of the harmonic series from last time. Some of you are still bothered that the sum

$$1 + 1/2 + 1/3 + 1/4 + \dots$$

can diverge even though the terms get smaller and smaller.

When something like

$$1 + 1/4 + 1/9 + \dots$$

converges.

Sum a thousand terms of the former, you get something like 7.5. Sum a million terms, you get 14.39. A billion terms, get 21.3. On the other hand, sum the first thousand terms of the latter, get 1.6439. Sum the first million, you get 1.6449. The first ten million, also 1.6449. The first one will just keep on growing. The second will converge. But of course I haven't *proved* these facts yet. Now we will.

Draw the picture of the graph of $y = 1/x$, along with a step function. FROM HERE ON TRY NOT TO ERASE.

Now. I want to prove to you that the limit

$$\sum_{i=1}^{\infty} 1/n$$

does not exist. So: (and here is a common ploy in mathematics) I will assume that it *does* exist and hope that a contradiction ensues.

So suppose it does exist. First, a less formal argument. Observe that the limit above is the "area under the step function" $g(x)$. Now we can't talk about that as an integral, because we haven't defined integral of non-continuous functions. But observe that this is the same situation as the comparison theorem. Our assumption is that there is a "finite area" under the step function. Whence also under $f(x)$. But this contradicts our earlier contradiction, Q.E.D.

More formally. Suppose the limit

$$\sum_{i=1}^{\infty} 1/n = L.$$

If it exists, it has to be equal to *something!* Now choose $B > L$. Then observe that

$$\int_1^n f(x)dx < \sum_{i=1}^{n} 1/n < B$$

So let $h(b) = \int_1^b f(x)dx$. We have shown that

- $h(b) < B$ for all $b \in [1, \infty]$.
- $h(b)$ is an increasing function of b. (Because $f(x) > 0$.)

Now it follows from Monotone Convergence that $h(b)$ converges as $b \to \infty$. So

$$\int_1^\infty f(x)dx$$

exists. Contradiction!

Emphasize that the way mathematicians actually work is to convince themselves first, via an argument like the "picture" we drew first. Subsequently, we try to "formalize" our thought process by means of a more precise proof. Both steps are crucial.

Theorem (Integral test) Suppose $f(x)$ is a continuous, decreasing function defined on $[1, \infty]$ with $f(x) > 0$ for all $x \in [1, \infty]$. Let $a_n = f(n)$.

If $\int_1^\infty f(x)dx$ diverges, $\sum_{n=1}^\infty a_n$ diverges.

We used this theorem above in case $f(x) = 1/x$. The proof is exactly as above. We

Challenge:

1. Where in my proof above did I use the fact that $f(x) > 0$? Would the theorem still hold without that assumption?
2. Where in my proof above did I use the fact that $f(x)$ was a decreasing function? Would the theorem still hold without that assumption?
3. Would the theorem still be true if I replaced the interval $[1, \infty]$ with the interval $[a, \infty]$?
4. Make the same assumptions as above. Is it true that

   If $\int_1^\infty f(x)dx$ converges, $\sum_{n=1}^\infty a_n$ converges.

This should take $20 - 25$ minutes. Notes to myself, since I haven't executed such a long groupwork before. Tell them beforehand that I'll be circulating and they should feel free to ask me questions when I come by. That they should work on whichever of these problems strike their fancy and are not expected to solve them all within the class period. That our goal is just to experience the process of wrestling with these problems.

After that, show as an example that the sum of $1/n^2$ converges, or even $1/n^3$ or $1/n^4$. Give values for the first and third of these, comment that the second was only recently shown to be irrational (I think.) The point being that infinite sums can be very mysterious.

We'll do more examples next time, when we have more technique under our belt. For now, if there's time, talk about estimating the *value* of a sum.

Ex: Estimate

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

First of all, observe that

$$\int_1^{\infty} \frac{1}{1+x^2} dx$$

is convergent, so by the integral test the sum above really does exist. There, I did some of your homework for you.

But suppose, drunk with success, we clamored to know what the value of this sum actually was? Your first impulse might be to ask your computer to sum a hundred terms. You get 1.0667. Try a thousand terms, get 1.0757. A hundred thousand, get 1.07666. (Satan!) A million, get 1.07667. You'd certainly be inclined to believe that the sum converged, and that the first few digits were 1.0766.... You've now *proven* that the sum converges. How could you *prove* that the sum lies within certain bounds? (This is really a question of, how much confidence do you need? I feel 99.9 percent confident that those are the right digits, but if I were building a bridge, I'd want to prove it.)

How to prove it: Draw the function and the step function again. Point out that the difference between our partial sum and the entire sum is the area of the step function starting at $1,000,001$. Write "what we want" is "partial sum" $+\sum_{1000001}^{\infty} 1/(1+n^2)$. Now move the step function over and show that that area is less than

$$\int_{1000000}^{\infty} \frac{1}{1+x^2} dx.$$

Be prepared to spend time on this–it may be hard. But once we do it, observe that the latter integral can be calculated to be

$$\pi/2 - arctan(1000000) = 10^{-6}$$

according to my computer. So the error is at most one millionth, and indeed these first few digits are *provably* correct.

Section 10.5: Comparison test

Don't forget to *start* by discussing the affirmative answer to the fourth question above.

If $f(x)$ is a continuous decreasing function defined on $[1, \infty]$, with $f(x) > 0$, and $a_n = f(n)$, then

If $\int_1^\infty f(x)dx$ converges, $\sum_{n=1}^\infty a_n$ converges.

Today we're going to talk about the comparison test–the comparison test and the integral test are our two most powerful tools for distinguishing convergent from divergent series.

Ex:(integral test) Consider $f(x) = 1/x^2$. Observe that

$$\int_1^\infty 1/x^2 dx = 1$$

whence $\sum^i nfty_{n=1} 1/n^2$ also converges.

Theorem: (Comparison Test) Let $(a_n), (b_n)$ be series such that $0 \leq a_n \leq b_n$ for all sufficiently large n.

If $\sum_{n=1}^\infty b_n$ converges, then so does $\sum_{n=1}^\infty a_n$.

If $\sum_{n=1}^\infty a_n$ diverges, then so does $\sum_{n=1}^\infty b_n$.

Ex:

$$\sum_{n=1}^\infty 1/(n^2+1)$$

converges.

This example uses $a_n = 1/(n^2+1), b_n = 1/n^2$. Notice that we could *also* carry this out via the integral test, seeing that the antiderivative converges to π_2. Now drawing the picture, we see that the sum must converge to something less than π_2. But what? Use discussion from previous page.

Ex:

$$\sum_{n=1}^\infty 1/(2n^2 - 3/2)$$

Now it is not true that

$$1/(2n^2 - 3/2) < 1/n^2$$

for all n, because it's not true at $n = 1$. But it is true thereafter. So this sum converges.

Ex:

$$\sum_{n=2}^{\infty} 1/(n^2-1)$$

We'll try the same trick again! Unfortunately, setting $a_n = 1/(n^2-1)$ and $b_n = 1/n^2$ does not work. Are we stuck?

No! Because note that

$$1/(n^2-1) < 1000/n^2.$$

How do I know that? It just says

$$\begin{aligned} n^2-1 &> n^2/1000 \\ 999n^2/1000-1 &> 0 \\ n^2 &> 1000/999 \end{aligned}$$

And this last thing is certainly true for all n bigger than 2. So letting $a_n = 1/(n^2-1), b_n = 1000/n^2$, we can use the comparison theorem.

But note that this wouldn't work for $a_n = 1/n$. Even if we had $a_n = 1/n, b_n = 1,000,000/n^2$, the a_n's would eventually "win."

Another way: carry out the indefinite integral.

Another way: use the partial fraction decomposition

$$1/(n^2-1) = (1/2)/(n-1) - (1/2)/(n+1)$$

to show that the sum telescopes to $3/4$.

Ex: Show that

$$\sum_{n=1}^{\infty} \sqrt{n+5}/n^3$$

converges. What to compare to? We'd like to compare to $1/n^3$. But we can't. Because this guy is bigger than $1/n^3$. So can we compare to c/n^3? Still no.

Are we stuck? No. Instead, make the comparison

$$\sqrt{n+5}/n^3 < n/n^3 = 1/n^2.$$

This is not true for *all* n, but it is true for n larger than 2.

Limit-comparison tests

The following theorem gets at some of the main ideas above.

Theorem. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms, and suppose that $\sum_{n=1}^{\infty} b_n$ has positive terms.

THEN

If $\lim_{n\to\infty} b_n/a_n$ exists, then $\sum_{n=1}^{\infty} b_n$ converges.

Suppose $\sum_{n=1}^{\infty} a_n$ is a divergent series with positive terms, and suppose that $\sum_{n=1}^{\infty} b_n$ has positive terms.

THEN

If $\lim_{n\to\infty} b_n/a_n$ exists and is not equal to 0, then $\sum_{n=1}^{\infty} b_n$ diverges.

See how this works for us? For instance, in the above example, we just say, let $a_n = 1/n^2$, and $b_n = \sqrt{n+5}/n^3$. Then the limit of b_n/a_n is the limit of $\sqrt{n+5}/n$, which exists and equals 0.

Why is this true?

First part: if the limit of b_n/a_n exists, then b_n/a_n can't grow without limit. In particular, there is some B such that $b_n/a_n < B$ for all n. But then $b_n < Ba_n$. Now use comparison on the series

$$\sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} Ba_n$$

Second part: left as contemplation.

Ex: Show that $\sum_{n=1}^{\infty} n/5^n$ converges.

We could compare with $1/5^n$, but that won't help because our thing is better. Instead, compare with, say, $a_n = 1/4^n$. And let $b_n = n/5^n$. Then $b_n/a_n = n4^n/5^n = n/(1.25)^n$. The limit of this is 0 as $n \to \infty$. (This fact is taken as given in the book) So by the limit-comparison test, the sum of b_n converges.

Section 10.6: Ratio and Root Tests

- No calculators on midterm.

OK. Today: the first thing I will do is go over the example of approximating the integral of $1(1+x^2)$, as discussed in the notes for section 10.4. Then I will do the limit-comparison test, with an example. Note that this is a more advanced version of the comparison test:

> If the comparison test works, the limit-comparison test will work.

The ratio test

Recall that a geometric series

$$a + ar + ar^2 + ar^3 + \dots$$

converged if and only if $|r| < 1$. Where r is the common ratio between terms. Each term is r times the last.

Theorem:(Ratio Test) Let $a_1 + a_2 + \dots$ be a series with $a_n > 0$.

If $\lim_{n\to\infty} a_{n+1}/a_n < 1$, the series converges. If $\lim_{n\to\infty} a_{n+1}/a_n > 1$, the series diverges. (If $\lim_{n\to\infty} a_{n+1}/a_n = 1$, no information)

For example,

$$\sum_{n=1}^{\infty} 1/(2^n - 1) = 1 + 1/3 + 1/7 + 1/15 + ...$$

You probably think that this converges, and you are right. Let them give some ideas other than ratio test for proving that it converges. Then say, well, observe

$$a_{n+1}/a_n = (2^n - 1)/(2^{n+1} - 1) = (1/2) - 1/2(2^{n+1} - 1)$$

which converges to $1/2$ as $n \to \infty$.

A common use of the ratio test is to deal with series involving *factorials*. Recall

$$n! = n(n-1)(n-2)\dots 1.$$

and we define $0! = 1$. (More justification for this later.)

Suppose we were going to study

$$\sum_{n=0}^{\infty} 1/n!.$$

which is on your homework.

Let's think of the methods we have so far.

- nth term test: Indeed, the terms approach 0, so we get no information.
- Integral test: we don't know how to define a function $1/x!$ which would be defined on all real numbers. (Although it can be done...!)

- Limit-comparison test: This powerful test should do the trick. What to compare it to? Try $b_n = 1/n^2$. Now

$$a_n/b_n = n^2/n! = n \cdot n/n(n-1)(n-2)\dots 1$$

and you probably believe this approaches 0 as $n \to \infty$. But you might have trouble proving it. Maybe you would say $n^2 \leq 2n(n-1)$ for all $n > 1$, so

$$a_n/b_n \leq 2/(n-2)(n-3)(n-4)\dots 1.$$

and so you'd have shown

$$\lim_{n\to\infty} a_n/b_n = 0$$

and by the limit-comparison test, $\sum a_n$ converges.

With our ratio test, though, it's easier– observe that

$$a_{n+1}/a_n = n!/(n+1)! = n(n-1)(n-2)\dots/(n+1)n(n-2)\dots = 1/n+1.$$

And so evidently

$$\lim_{n\to\infty} a_{n+1}/a_n = 0$$

and so the series converges.

Now what about

$$\sum_{n=0}^{\infty} 10^n/n!$$

Recall that we showed in the homework that the *terms* of this series converged to 0. Does the sum converge? We're looking at

$$1 + 10/1 + 100/2 + 1000/6 + 10000/24 + 100000/120 + \dots$$

Again, the other tests may be hard to apply. But note that

$$a_{n+1}/a_n = (10^{n+1}/(n+1)!)/(10^n/n!) = 10^{n+1}n!/10^n(n+1)! = 10/(n+1)$$

and again, we see that

$$\lim_{n\to\infty} a_{n+1}/a_n = 0$$

so $\sum_{n=0}^{\infty} 10^n/n!$ converges.

The root test

Another related test is the root test. It is also motivated by the geometric series

$$ar + ar^2 + ar^3 + \dots$$

with $a_n = ar^n$. (Note that I wrote this in a different form from that which I usually use–but all I did was leave off the first term, so that is OK!) Now note that

$$(a_n)^{1/n} = (ar^n)^{1/n} = a^{1/n}r$$

so that

$$\lim_{n\to\infty} (a_n)^{1/n} = r$$

Recall that this series converges if and only if $|r| < 1$. Likewise,

Theorem (Root test) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.

- If $\lim_{n\to\infty} a_n^{1/n}$ exists and is less than 1, $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim n \to \infty a_n^{1/n}$ "is ∞", or if it is greater than 1, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim n \to \infty a_n^{1/n}$ is equal to 1, **no information.**

  When to use it: when you have "something to the n". After all, that's the only time taking an nth root will simplify something.

Ex: Show that

$$\sum_{n=2}^{\infty} 1/(\log n)^n$$

converges.

This follows immediately from the root test.

Compare:

$$\sum_{n=2}^{\infty} 1/(\log n)^{n-1}$$

Ask: does this still converge? Why?