Section 8.8: Improper Integrals

One of the main applications of integrals is to compute the areas under curves, as you know. A geometric question. But there are some geometric questions which we do not yet know how to do by calculus, even though they appear to have the same form. Consider the curve $y = 1/x^2$. We can ask, what is the area of the region under the curve and right of the line $x = 1$? We have no reason to believe this area is finite, but let's ask.

Now no integral will compute this–we have to integrate over a bounded interval. Nonetheless, we don't want to throw up our hands.

So note that

$$\int_1^b (1/x^2)dx = (-1/x)|_1^b = 1 - 1/b.$$

In other words, as b gets larger and larger, the area under the curve and above $[1, b]$ gets larger and larger; but note that it gets closer and closer to 1. Thus, our intuition tells us that the area of the region we're interested in is exactly 1. More formally:

$$\lim_{b\to\infty} 1 - 1/b = 1.$$

We can rewrite that as

$$\lim_{b\to\infty} \int_1^b (1/x^2)dx.$$

Indeed, in general, if we want to compute the area under $y = f(x)$ and right of the line $x = a$, we are computing

$$\lim_{b\to\infty} \int_a^b f(x)dx.$$

ASK: Does this limit always exist? Give some situations where it does not exist. They'll give something that blows up. Ask: suppose the function is bounded. Then does the limit always exist? Then say: suppose the limit of the function is 0. Then does the limit always exist?

An *improper integral* is an expression of the form

$$\int_a^\infty f(x)dx.$$

It is said to be *convergent* if the limit

$$\lim_{b\to\infty} \int_a^b f(x)dx.$$

exists, and *divergent* otherwise. In the first case, we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

Likewise,

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

If the integral goes to ∞ on both sides, write it as a sum of two improper integrals $\int_0^\infty$ and $\int_{-\infty}^0$.

If we haven't already done it, give

$$\int_0^\infty \sin x dx$$

as an example of an integral that doesn't converge, even though it doesn't blow up. Draw the picture and point out that here, fortunately, our geometric intuition concurs that it does not make sense to talk about "the area of the shaded region." Ideally, our mathematical theories should conform to our geometric intuitions–then in conditions of extremity the causality should work the other way...

Main problem of the day. How do we distinguish between improper integrals which converge and those which diverge?

Ask for a vote on the following examples. Give them 2 minutes to think.

1. $\int_1^\infty x^4 dx$
2. $\int_1^\infty 1/x^4 dx$
3. $\int_1^\infty 1/1 + x^4 dx$
4. $\int_0^\infty e^x dx$
5. $\int_{-\infty}^0 e^x dx$
6. $\int^\infty)_{-\infty} e^x dx.$

Now let's develop some techniques to deal with these guys.

The comparison technique

This is the number one best way to show that an integral converges.

Theorem. Let f be a continuous function defined on $[a, \infty]$ such that $f(x) \geq 0$ for all $x \in [a, \infty]$. Suppose there is another function g such that $f(x) \leq g(x)$, and such that

$$\int_a^\infty g(x)dx$$

converges. Then

$$\int_a^\infty f(x)dx$$

converges, and moreover

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx.$$

This makes sense. Draw the picture. If the area under g is finite, so is the area in the subregion under f.

Let's show how a proof would go. Define

$$h(b) = \int_a^b f(x)dx.$$

Then $h(x)$ is an increasing function. We want to show that $h(b)$ has a limit as $b \to \infty$ and that this limit is less than $\int_a^\infty g(x)$. We have

$$h(b) = \int_a^b f(x)dx \leq \int_a^b g(x)dx \leq \int_a^\infty g(x)dx.$$

So draw this picture, and now recall there's a theorem that an increasing bounded function has a limit. "Everything that rises must converge," as they say. Note that the condition that h be increasing–which is the same as the condition that f be positive–is absolutely essential. Draw a picture.

Ex: Consider Nate's integral

$$\int_0^\infty dx/(x^5 + 3x^4 + 4x^2 + 5x + 7)^{7/8}.$$

Whether or not Nate can do this integral, or even whether or not I can do it, I can tell you that the improper integral above is convergent. How? Well, I'll use two steps. First of all, note that I can write it as

$$\int_1^\infty dx/(x^5 + 3x^4 + 4x^2 + 5x + 7)^{7/8} + \int_0^1 dx/(x^5 + 3x^4 + 4x^2 + 5x + 7)^{7/8}$$

where the latter integral is perfectly well-defined, so does not affect convergence. Draw the picture. Then even say, look,

$$\int_0^b f(x)dx = \int_1^b f(x)dx - \int_0^1 f(x)dx$$

and the latter thing is a constant, so can be passed through the limit. Anyway. The point is that Nate's function is smaller than $1/(x^5)^{7/8} = 1/x^{35/8}$. And we find that

$$\begin{aligned}\int_1^\infty dx/x^{35/8} &= \lim_{b\to\infty} \int_1^b dx/x^{35/8} \\ &= \lim_{b\to\infty} -(8/27)/x^{27/8}|_1^b \\ &= \lim_{b\to\infty} -(8/27)/b^{27/8} + 8/27 \\ &= 8/27.\end{aligned}$$

It follows from the theorem that Nate's integral exists and is at most 8/27.

Remark: The theorem also works for integrals to $-\infty$, as will everything I say today. From now on, I probably won't mention it.

Note that we can turn this theorem on its head, and use it as a criterion for divergence.

Theorem. Given f, g as above. If

$$\int_a^\infty f(x)dx$$

diverges, so does

$$\int_a^\infty g(x)dx.$$

Ex: The integral

$$\int_2^\infty x^2/(x^3-1)dx$$

diverges, because the function is greater than $1/x$, and we have already seen that this integral diverges.

ASK: Why did I have 2 as my lower bound there? Why not 1 or 0?

This leads us to:

Unbounded functions

Draw the graph of $x^2/(x^3-1)$. We might as well ask: what is the area between 0 and 2? This makes geometric sense, even though we haven't defined integrals over regions where the function is unbounded. Well, what *does* make sense? We can write

$$\int_0^b x^2/(x^3-1)dx$$

and

$$\int_c^2 x^2/(x^3-1)dx$$

(draw the picture) and the intuition is that, as b gets close to 1 from below, and c gets close to 1 from above, the sum of the two integrals above should get closer and closer to the actual area, if it exists. To solve this, note that the expression has an indefinite integral $(1/3)\log|x^3-1|$. So the expressions above turn into

$$(1/3)\log|-1| - (1/3)\log|b^3-1| = -(1/3)\log|b^3-1|$$

while the latter becomes

$$(1/3)\log 7 - (1/3)\log|c^3-1|.$$

Neither of these approaches a limit as b, c nears 1. We conclude that the integral is divergent. However, it is possible for the integral of a function over an interval to converge even when the function is not bounded on that interval. See example 5 in your book.

Absolute value test

In the comparison test, we deal entirely with functions which take on positive values. In life, that isn't always the way of it. What if our function takes on both positive and negative values?

Thoughts: On the one hand, we'd think convergence was *more* likely, because the positive and negative parts would cancel each other out. On the other hand, we can have problems with oscillation, as we've seen, so maybe it's harder. We at least have

Theorem. Let f be a continuous function defined on $[a,\infty]$. Suppose that

$$\int_a^\infty |f(x)|dx = L$$

is a convergent indefinite integral. Then

$$\int_a^\infty f(x)dx$$

converges and its value lies between L and $-L$.

Proof by picture, and previous theorem. We show by the previous theorem that the "positive part" A and the "negative part" B both converge, and moreover that $A < L$ and $B > -L$, and since $A + B$ is the desired integral, we're done.

Remark: We've implicitly used the fact that if an integral splits into the sum of two integrals, each of which converge, then the whole integral converges. Why?

Rational functions

If there's time (which there won't be), ask: for which $P(x)$ is it true that

$$\int_{-infty}^\infty dx/P(x)$$

converges? What general results can we come up with?

Prompt them to come up with some examples from today's class. See if we can get to a general idea of what must happen. Especially see if I can get anyone to suggest a general class of polynomials making this converge. Leave it as a contemplation.

Section 10.2: Sequences.

- Check out http://forum.swarthmore.edu/dr.math/.

Suggest that they read Section 10.1, as it'll motivate what we're doing more. For now, hit them with the following.

$$e = 1 + 1/2 + 1/3! + 1/4! + 1/5! + \dots$$

This kind of "infinite sum" will be of crucial importance for us in the weeks to come and we had better get ourselves ready to understand what it means.

But we've got to walk before we run.

An infinite sequence is just a list

$$a_1, a_2, a_3, \dots$$

For instance,

$$1, 2, 3, 4, 5, \ldots$$

or

$$1, 1/4, 1/9, 1/16, \ldots$$

or

$$1, 1/2, 1/6, 1/120, \ldots$$

which we will write for shorthand as

$$n, 1/n^2, 1/n!.$$

Definition: We say that L is the *limit* of the sequence $a_0, a_1, \ldots$ if, for all any interval $[a, b]$ containing L, the terms of the sequence eventually lie entirely within $[a, b]$.

Ex: For the second series above, 0 is a limit. Because if you draw any interval, say, $[-0.01, 0.01]$. Then don't you agree that after a given point, all the elements of the sequence lie inside the interval? In particular, a_10 lies in this interval, and so do all further a_n.

Ex: For the third series above, 0 is also a limit.

Ex: The first series above does not have a limit.

Theorem: No sequence can have two different limits.

Proof. Suppose L and L' were limits for the sequence. Then choose intervals $[a, b]$ containing L and $[c, d]$ containing L' which are so small that they do not intersect. (Draw this.) Then eventually all the terms of the sequence have to be in $[a, b]$. But then they can't be in $[c, d]$! Contradiction.

This fact was obvious anyway from your intuitive notion of limit. But then again, if I asked you to prove it, you'd throw up your hands, and say, "it's obvious!" You have to have a precise definition in order to prove anything. Note that even here we relied on "obviousness" when choosing our "small intervals." Exercise: finish the proof by proving that such small intervals really do exist.

A sequence that has a limit is called *convergent* and a sequence without a limit is called *divergent.*

Vote on:

1. $(0.5)^n$;
2. 5^n;

3. 1^n;

4. $(-1)^n$;

5. $(-1^n)/n!$;

6. $(2^n)/n!$;

If there's some controversy, any at all, divide into twos, discuss 3 minutes, vote again.

Theorem: (Monotone Convergence) Suppose a_n is a non-decreasing sequence, and is bounded by B; that is, $a_n < B$ for all n. Then a_n has a limit, which is less than B.

This is a deep, deep theorem. First of all, it seems obvious. Then you try to prove it and you get very confused. It's sort of like this. You want to prove something about limits. But you just throw up your hands and say, "it's obvious!" What's the problem? That you haven't precisely defined what a limit is-you just know. Not good enough. When trying to do a problem like this, you find that you haven't precisely defined what real numbers are. And in fact this had to wait until Dedekind in 1858– to be precise on 24 November 1858.

Likewise, this works upside down; if a_n is a non-increasing sequence bounded below, it has a limit. So for instance, for the last guy above, show that it eventually decreases, and since it is bounded below by 0, it has a limit. In fact, the limit is 0.

To see this, write

$$2^n/(n!) = 2 * 2 * 2 * 2 * \ldots * 2/1 * 2 * 3 * 4 * \ldots * n$$

Now we see that this is less than

$$2^n/(n!) = 2 * 2 * 2 * 2 * \ldots * 2/1 * 2 * 3 * 3 * \ldots * 3 = 2/1 * 2/2 * (2/3)^{n-2}$$

which evidently approaches 0. Draw the "squeeze in" picture. Note the similarity to the comparison theorem picture.

Section 10.3: Infinite series

- Tell them to put their section time and my name on paper. And STAPLE.

An infinite series is a special kind of infinite sequence. Namely, it is an infinite sum.

Examples:

- $1 + 1/3 + 1/9 + 1/27 + 1/81 + \dots$
- $1 - 1 + 1 - 1 + 1 - 1 + \dots$
- $1 + 1 + 1 + 1 + 1 + \dots$
- $1 + 1/4 + 1/9 + 1/16 + 1/25 + \dots$

The first case, for instance, refers to the infinite sequence

$$1, 4/3, 13/9, 40/27, \dots$$

and the third refers to

$$1, 2, 3, 4, 5, \dots$$

So

$$a_1 + a_2 + \dots = \sum_{i=1}^{\infty} a_i$$

is defined to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_n$$

if this exists. The $\sum_{i=1}^{n} a_n$ are called *partial sums.*

As always, the MAIN QUESTION is:

How do we tell a convergent series from a divergent one?

So let me make a point. It's not whether the things we're summing converge. Consider the third series above. The *things we're summing* are always 1. But the sum does not converge.

Ex: Geometric series.

These are series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

for some a, r. ASK: Which of the above series are geometric?

FACT:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n = a(1 - r^n)/(1 - r), r \neq 1$$

We can check this by multiplying. Example: $1+x+x^2 = (1-x^3)/(1-x)$, which you probably know as the factorization $(1-x^3) = (1-x)(1+x+x^2)$. So if we sum n terms of the sequence we get the above expression. Question: does this converge? Sit and let them think about it.

FACT: It converges if and only if $|r| < 1$, in which case it converges to $a/(1-r)$. So in particular, the first series above converges to $1/(1-1/3) = 3/2$.

BASIC IDEAS.

- (Monotone convergence theorem.) Suppose all the terms $a_1, a_2, a_3, \ldots$ are positive, and that there is some upper bound B such that $a_1 + a_2 + \ldots + a_n < B$ for all n. Then $a_1 + a_2 + \ldots$ converges.

  Ex:: $1 + 1/10 + 1/10^4 + 1/10^9 + 1/10^16 + \ldots = \sum_{i=1}^{\infty} 1/10^(i^2)$

  Write the partial sums out as decimals:

  $$\begin{aligned} a_1 &= 1 \\ a_1 + a_2 &= 1.1 \\ a_1 + a_2 + a_3 &= 1.1001 \\ a_1 + a_2 + a_3 + a_4 &= 1.100100001 \end{aligned}$$

  Clearly these are always less than 2. So by the theorem this series converges.

  Natural complaint: But we already *know* this sequence converges! Because it converges to 1.1001000010000001.... But then you have to ask yourself, why do we believe that such a number exists? Now we are dipping perilously close to "foundations," something that confused mathematicians for a long time.

  In this connection, Liz asked me last time "Is this improper integral we get *really* the area or is it just something that gets closer and closer to the area?"

  Similarly, one might ask, are we saying that 4/3 is *really* the value of the sum above, or only that the sum gets closer and closer to it?

  In this connection, quote Hardy, from *Divergent Series* (1948).

> ...it does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by X we *mean* Y.' ... it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1-1+1-1+\dots$ but 'What *is* $1-1+1-1+\dots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.

- Suppose the series $a_1 + a_2 + a_3 + \dots$ converges. Then the sequence $a_1, a_2, a_3, \dots$ converges to 0.

  We'll not prove this, though the book gives a slick proof. For us, consider the *contrapositive.*

  Suppose the sequence $a_1, a_2, a_3, \dots$ does not converge to 0. Then the sequence $a_1 + a_2 + a_3 \dots$ converges.

  We'll do the case where $a_1, a_2, a_3, \dots$ converges to $L \neq 0$. Then after a while the sequence

  $$a_1 + a_2 + a_3 + \dots + a_1 00 + a_1 01 + \dots$$

  is well-approximated by

  $$a_1 + a_2 + a_3 + \dots + L + L + \dots$$

  and evidently this sequence does not converge.

  The *converse* of this statement would be

  Suppose the sequence $a_1, a_2, a_3, \dots$ converges to 0. Then the series $a_1 + a_2 + a_3 + a_4 + \dots$ converges.

  True or false? Vote: then split into pairs, discuss, come back to me.

Ex: The harmonic series.

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \dots$$

The terms converge to 0. Nonetheless, the sum diverges.

To prove this, observe that we can split the terms up like

$$(1 + 1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8)+$$
$$(1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1$$

and the terms within each parenthesis add up to at least 1/2. So the partial sums are unbounded.

Again, remember that the *sequence* of partial sums is

$$1, 1 + 1/2, 1 + 1/2 + 1/3, 1 + 1/2 + 1/3 + 1/4$$

Now note that this series indeed diverges, but it diverges *much more slowly* than something like $1 + 1 + 1 + 1 + \ldots$. In higher math, once we determine that a sequence converges or diverges, we like to ask, HOW QUICKLY?

According to our calculation, it takes *at most*

- 2 terms to get to 1/2;
- 4 terms to get to 1;
- 8 terms to get to 3/2;
- 16 terms to get to 2;
- 32 terms to get to 5/2;
- 64 terms to get to 3;
- 4^k terms to get to k;

Actual calculation shows it takes

- 1 term to get to 1;
- 4 terms to get to 2;
- 11 terms to get to 3;
- 31 terms to get to 4;
- 83 terms to get to 5;
- 227 terms to get to 6;
- 616 terms to get to 7;

- 1674 terms to get to 8...

In other words, our argument shows that "number of terms necessary to get to k is $\leq 4^k$. This suggests that maybe that number grows exponentially! And indeed, we find that by experimental data, it is very close to the truth that "number of terms necessary to get to k is very close to e^k."

Another way to say it is that

$$1 + 1/2 + 1/3 + \ldots + 1/e^k \sim k.$$

or

$$1 + 1/2 + 1/3 + \ldots + 1/n \sim \log n$$

Does this look like anything? Draw the curve, the step function, the integral. And this prepares them for next time.