Section 7.4: Rational Functions

- Office hours today after I teach, at noon. Next week, the time will become 2:30 pm on Tues.
- The Missouri Club-you can talk about the week's homework with loving graduate students. Meets Sunday nights, 8-11 pm, in McCosh 34. They say, "please show up at 8." See their web site, easy to find.
- Electra, at the McCarter.
- Hand out "elementary functions" handout from Courant.

A *rational function* is simply the quotient of two polynomials, like say

$$\frac{x+1}{x^3+x}.$$

These kinds of functions come up all the time, and the happy fact is, we have a pretty well-worked out way to integrate them. ASK for examples of rational functions we already know how to integrate. If they don't come up with 1/x, ask, is the integral of a rational function always a rational function?

What about this one I just wrote up? How can we do it? Will substitution work? Maybe not.

But here's something. Write

$$\frac{x+1}{x^3+x} = \frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{x^2+1}$$

(show this is correct) and integrate each piece, ending up with

$$\log|x| + \arctan x - (1/2)\log|x^2+1|.$$

This is called *the method of partial fractions*, because I split a rational function (fraction) up into a sum of simpler rational functions. This naturally brings up two questions.

1. Which rational functions are simple enough for us to integrate?
2. Which rational functions can be split up into sums of "simple" rational functions?

3. How do we figure out how to do it?

Today, we'll talk about question 1.
Ex: We can do any polynomial.
Ex: We can do

$$\int \frac{1}{ax+b} dx$$

.

Namely, we just use the substitution $u = ax + b$, in which case $x = (u - b)/a$, so $dx = du/a$, and we end up with

$$\int (1/a)(du/u) = (1/a)\log|u| = (1/a)\log|ax+b|.$$

Easy enough.

Ex: Another one we can do is $\int 1/(1+x^2)dx$. As remarked above, it is $\arctan x$. But maybe cooler is that we can do

$$\int \frac{1}{ax^2+bx+c} dx.$$

So the goal is: how can we make the denominator look more like x^2+1? This is a good time for a group exercise.

Challenge: Given the reciprocal of a quadratic as above, can you transform it into something like $\int 1/(1+u^2)du$ via a judicious substitution? For which quadratics is this possible? Examples to think about:

$$\int \frac{1}{x^2+9} dx, \int \frac{1}{9x^2+1}, \int \frac{1}{x^2+4x+5} dx$$

Split into groups of four, give seven minutes to think on this, then regroup and check what progress has been made.

See if they thought of $u = (\sqrt{a})x$. That's something. That gets us to

$$\int (1/\sqrt{a}) \frac{1}{u^2 + (b/\sqrt{a})u + c} du.$$

And then the key is to figure out a substitution that'll make the denominator look like v^2 plus a constant. And this comes down to "completing the square." We observe that

$$(u+\alpha)^2 = (u^2 + 2\alpha u + \alpha^2)$$

so for example

$$u^2 + 4u + 4, u^2 + 12u + 36, u^2 + 2000u + 1000000$$

are all squares. Likewise,

$$u^2 + (b/\sqrt{a})u + (b^2/4a) = (u + b/\sqrt{a})^2.$$

So subbing in $v = (u + b/\sqrt{a})$, we get

$$\int (1/\sqrt{a}) \frac{1}{v^2 + (c - (b^2/4a))} dv$$

which is just about what we want! The only thing to fix now is that constant coefficient, which (as we might have figured out in group work) can be done by substituting $w = \sqrt{4a/(b^2 - 4ac)}v$, so $dv = \sqrt{(b^2 - 4ac)/4a}dw$ and we end with

$$\int (\sqrt{b^2 - 4ac}/2a) \frac{1}{w^2 + 1} dw$$

which we can finally do. Only problem: what if $b^2 - 4ac$ is negative?

If there's time, which there won't be, talk about

$$\int x/(x^2 - 1).$$

1 Section 7.5: Integration by Partial Fractions

Note to myself: I missed a lecture this week due to Yom Kippur, so I have notes only for one lecture, not two, on partial fractions!

Remind them they've got to read the book–use the examples, even (especially) the ones I don't talk about.

Main theme: How do I split up a rational function? Recall I wrote

$$\frac{x+1}{x^3 + x} = \frac{1}{x} + \frac{1}{x^2 + 1} - \frac{x}{x^2 + 1}.$$

How did I know to do that? Suppose I'm given a rational function $P(x)/Q(x)$. **Step 1** is to factor the denominator. Here we factor

$$x^3 + x = x(x^2 + 1).$$

Now step 2 is that we usually know what form we *expect* the partial fraction to take. In this case, I expect it to be of the form

$$\frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

At this point, don't say, "how do I know to expect this?" Say instead it is a gift from the mathematical past. Now the question is, how do I choose these constants correctly? Well, writing a common denominator, I get

$$\begin{aligned} & \frac{Ax^2+A}{x(x^2+1)} + \frac{Bx^2+Cx}{x(x^2+1)} \\ = & \frac{(A+B)x^2+Cx+A}{x^3+x}. \end{aligned}$$

We conclude that $A = 1, C = 1, A + B = 0$. Conclude $B = -1$, which gives the formula above. (We have used the "comparison of coefficients" method–the book describes another method, the "substitution" method, which you may also use if you prefer it.)

So the question is, how do we know what to expect?

Answer. Suppose the denominator factors as

$$(ax+b)(cx+d)...(jx+k)(lx^2+mx+n).....(wx^2+yx+z).$$

And suppose for the moment that the numerator has smaller *degree* than the denominator; that is, the power of x is greater in the denominator than in the numerator.

Then we expect the partial fraction to take the form

$$\frac{A}{ax+b} + \frac{C}{cx+d} + \ldots + \frac{Lx+M}{lx^2+mx+n} + \ldots + \frac{Wx+Y}{wx^2+yx+z}.$$

Ex: How would I expect a partial fraction for

$$\frac{1}{(x^2-1)(x^2+1)(x^2+9)}$$

to look?

Ex:

$$\int \frac{x^3+3x}{x^4+6x^2+5} dx.$$

Get them to give the partial fraction form

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+5}$$

Then multiply out both sides:

$$\begin{aligned}(Ax+B)(x^2+5) &= (Ax^3+Bx^2+5Ax+5B)\\(Cx+D)(x^2+1) &= (Cx^3+Dx^2+Cx+D)\end{aligned}$$

and conclude that

$$(A+C)x^3+(B+D)x^2+(5A+C)x+(5B+D)=x^3+3x$$

So $B+D=5B+D=0$; subtracting, conclude $B=D=0$. We're left with

$$\begin{aligned}A+C &= 1\\5A+C &= 3\end{aligned}$$

Conclude that $A=1/2$, $C=1/2$. So we have

$$\int(1/2)\frac{x}{x^2+1}+(1/2)\frac{x}{x^2+5}dx$$

or

$$(1/4)\log|x^2+1|+(1/4)\log|x^2+5|.$$

Now observe that we could have directly computed the integral as

$$(1/4)\log|(x^2+1)(x^2+5)|$$

from the outset! Hope someone is impressed.

Now some natural questions should arise. Actually, it would be good to SOLICIT questions from the audience, write them up on one part of the board, then treat them in order. That is, ask "have I or have I not convinced you that given any $P(x)/Q(x)$, I can integrate it?" In fact, if time is long, make groups and have them think of some issues left to treat.

- What if there is a factor of degree greater than 2? Answer: an amazing fact–*The Fundamental Theorem of Algebra* tells us that *any* polynomial can be factored into linears and quadratics! So no worries.

- When I use this method, will I always be able to find a solution in the A, B, et cetera? Take a vote. Then appeal to linear algebra– the beginning of the mathematical branch of *algebra*, which essentially asks: "which equations can be solved?"

- What if we have repeated roots downstairs? Answer: there's a method. The idea is that if I have

  $$\frac{P(x)}{(x-2)^3(x^2+1)^2}$$

  I write

  $$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}.$$

  Looks fun, huh? If time, which there won't be, do an example.

- What if the degree of the numerator is *not* smaller than the degree of the denominator?

  Ex: Compute

  $$\int \frac{x^4}{x-1}dx.$$

  In this case, we have to creak out our old friend, synthetic division. Work this out at the board, and show that

  $$x^4 = (x^3 + x^2 + x + 1)(x-1) + 1.$$

  Whence we are just integrating

  $$x^3 + x^2 + x + 1 + \frac{1}{x-1},$$

  which we can certainly do.

So I've achieved my stated goal of showing you how to integrate *any* rational function!