

Parametric Curves

First of all: the final homework is *not to be turned in* but I implore you to do it. Because “this will be on the test,” as they say.

What is a curve? We think of it as being given by an equation

$$y = f(x)$$

or maybe an implicit equation like

$$x^2 + y^2 = 4.$$

But there is another, very important method of describing a curve. We will talk about the famous *cycloid* curve. How does it go? Well, imagine a sparkler taped to a locomotive wheel of radius 1 meter. Draw the picture. How can we describe its motion? We are asking for:

$$(x(t), y(t)) : \text{position at time } t.$$

Well. Let's say the train is moving at 10 meters per second, so that the horizontal position of the center of the wheel at time t is $x = 10t$. Moreover, the vertical position is constant at 1. So the position of the center of the wheel is given by $(10t, 1)$.

But that's not what we want to find!

So how to describe the position of the sparkler? Well, at time t , the sparkler is at some angle $\theta(t)$. How to compute this? That is, how to compute the angular velocity? Well, how about this. In one second, the wheel rolls 10 meters. How many revolutions is this? Solicit answers. Ask for questions *to the person who answered*. OK. So after time t , the wheel has rolled forward $10t$ meters, which corresponds to $10/2\pi$ revolutions. Which is to say, an angle of 10 radians. In other words, $\theta(t) = -10t$.

Now draw the right triangle and conclude that

$$(x(t), y(t)) = (10t + \cos(-10t), 1 + \sin(-10t)) = (10t + \cos(10t), 1 - \sin(10t)).$$

So we have parametric equations! And we could solve for y in terms of x , but we'd get a mess.

DISADVANTAGE of solving for y in terms of x ; it's a mess.

ADVANTAGE: We know how to do calculus with curves $y = f(x)$.

So we have **GOAL:** Learn to do calculus on parametrized curves.

First step: how to find the slope? Well, we know

$$\text{slope} = dy/dx.$$

And remember from the chain rule that if y is a function of x , which is in turn a function of t , then

$$dy/dt = dy/dx dx/dt$$

or

$$dy/dx = (dy/dt)/(dx/dt).$$

Application: compute the slope of the cycloid curve at $t = 0$. Well, we get $dx/dt = 10 - 10 \sin(10t)$ and $dy/dt = -10 \cos(10t)$. So conclude

$$dy/dx = -\cos(10t)/(1 - \sin(10t))$$

which, evaluated at 0 is -1 . So that is the slope, which kind of makes sense.

Application: compute the area under one “arch” of the cycloid curve. This is a serious problem! First of all, we have to figure out the bounds of integration. So when does it touch the x -axis? Get the figures, at $t = \pi/20$ and then again at $t = 5\pi/20 = \pi/4$.

Now split into pairs and give them just 5 minutes to set up the integral. Presumably most of them will get

$$\int_{\pi/20}^{\pi/4} 1 - \sin(10t) dt = \pi/5.$$

But this is not correct. How can we see this? Well, think of it this way—the average height is about 1 meter off the ground, and the wheel has travelled 2π meters. So $\pi/5$ is an unreasonably small answer.

Instead, draw the partition of the t interval into $\pi/20 = t_0 < t_1 < \dots < t_n = \pi/4$. And draw the rectangles. Now the area of a rectangle between t_i and t_{i+1} is approximately

$$y(t_i)(x(t_{i+1}) - x(t_i))$$

and the total area is approximated by

$$\sum_{i=1}^n y(t_i)(x(t_{i+1}) - x(t_i)).$$

Note that this is NOT yet a Riemann sum! We have $x(t_{i+1}) - x(t_i)$, and what we want is $t_{i+1} - t_i$.

So let's see if we can analyze the ratio between these things:

$$\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}$$

And here is the point: as $t_{i+1} - t_i$ decreases, this becomes very close to $x'(t_i)$! So in one lecture we have used the definition of the derivative and the definition of the integral...! Anyway, we conclude that the sum is approximately

$$\sum_{i=1}^n y(t_i)x'(t_i)(t_{i+1} - t_i)$$

which converges, as the mesh gets small, to

$$\int_{\pi/20}^{\pi/4} y(t)x'(t)dt = \int_{\pi/20}^{\pi/4} (1 - \sin(10t))(10 - 10 \cos(10t))dt.$$

Multiplying out, get

$$10 \int_{\pi/20}^{\pi/4} 1 - \sin(10t) - \cos(10t) + \sin(10t) \cos(10t)dt$$

and this comes out to 2π , a much more reasonable (not to mention correct) answer.

Now pass out evaluation forms.

1 Arclength and surface area

We discussed last time that the right way to find arclength was to integrate

$$\int \sqrt{(dx)^2 + (dy)^2}.$$

In general, write

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

If we have a parametrized curve, ds/dt is the amount of distance travelled in some amount of time; in other words,

$$ds/dt = \text{speed}.$$

Ex: Consider the curve $x(t) = t \cos(t)$, $y(t) = t \sin(t)$.

Get them to plot some points on the curve, see what it looks like.

Q: What is the length of the curve between $t = 0$ and $t = 5$?

Length is given by

$$\int_{t=0}^{t=5} ds = \int_{t=0}^{t=5} \sqrt{(dx)^2 + (dy)^2}$$

Now $dx = x'(t)dt = (\cos(t) - t \sin(t))dt$, and $dy = y'(t)dt = (\sin(t) + t \cos(t))dt$. So

$$(dx)^2 + (dy)^2 = (\cos^2(t) - 2t \cos(t) \sin(t) + t^2 \sin^2(t) + \sin^2(t) + 2t \cos(t) \sin(t) + t^2 \sin^2(t))(dt)^2$$

which after you cancel is

$$(1 + t^2)(dt)^2.$$

So we've just got to integrate

$$\int_0^5 \sqrt{1 + t^2} dt = (1/2)[5\sqrt{26} + \log(5 + \sqrt{26})]$$

by integral table. In a moment we'll try to do one that doesn't require use of an integral table.

Q: What is the speed of the particle at time $t = 5$?

We just observe

$$ds/dt = \sqrt{1 + t^2} = \sqrt{26}.$$

A word about surface area. I won't spend time on this—I charge you with reading it. But here is the idea. Draw a picture of a surface of revolution. Comment that the change in surface area between time t and time $t + dt$ is just about $2\pi R ds$. So in general,

$$\text{surface area} = \int 2\pi R ds.$$

OK, that said, I want to do

Groupwork: What is the length of the parabola $y = x^2/2$ between $x = 0$ and $x = 5$?

Thank you very much and good night!