Section 8.1-4: Definite integrals and volumes.

This week, we switch gears, forget about infinite series and address the problem of using integrals to compute volumes and areas in three-dimensional contexts.

Section 8.1: This reminds you how definite integrals work. Section 8.2: this is a primer in drawing 3-dimensional pictures. I wouldn't mind you reading each of these

Ok. First, recall how we derive, say, the area between the parabola $y = x^2$ and the line $y = 16$. Draw the picture. Say, on the one hand we think of this as the sum of a lot of very narrow rectangles, where the rectangle at position x has height $16 - x^2$ and width dx. So we express the limit of the sums as

$$\int_{-4}^{4} (16 - x^2)dx.$$

In general, to compute the area of some region, we are looking at

$$\int_a^b h(x)dx$$

where $h(x) =$ "height of the cross-section at x."

Now suppose we are instead trying to understand the VOLUME of the figure bounded between the surfaces $z = x^2 + y^2$ and $z = 16$. Draw this paraboloid as best I can. How do I know it looks like this? Well, we can say z is the square of the distance from the origin..

Anyway, now make slices in the z-direction, like the counterhand at the deli slicing corned beef. We need to estimate the volume of each slice. Its width is dx. So the volume of the whole thing will be

$$\int_0^{16} A(z)dz$$

where now $A(z)$ is the *area* of the cross-section at z. Perhaps you already believe this–I'll go into a little more detail below.

The cross-section is not merely rectangular. In fact, it is a cylinder, of radius $r = \sqrt{z}$. So

$$A(z) = \pi r^2 = \pi z.$$

Now what does this tell us if we split the interval from 0 to 16 up into small intervals $0 = z_0 < z_1 < z_2 < \ldots < z_n = 16$? Well, the ith slice has thickness

$z_{i+1} - z_i$. And the volume of the slice is thus approximately the volume of the cylinder I draw, or

$$A(z_i) * (z_{i+1} - z_i).$$

So we are looking at the limit, as the slicing gets finer and finer, of

$$\sum_{i=0}^{n} A(z_i) * (z_{i+1} - z_i)$$

and this limit is the very definition of

$$\int_0^{16} A(z)dz.$$

So the total volume is

$$\int_0^{16} \pi z dz = \pi z^2/2|_0^{16} = 128\pi.$$

We have lots of choices as to how to carry out our slicing. One good way is to look at a section of the region over a very narrow RING in the $x - y$ plane. Draw the annulus outside radius $r = \sqrt{x^2 + y^2}$ and inside radius $r + dr$. What does this cross section look like? Prompt for answers. It is a cylinder, with radius r and height $16 - r^2$. So what is its surface area? One way is to imagine slicing it and unrolling. Then we get a rectangle with length $2\pi r$ and height $16 - r^2$. So we conclude

$$A(r) = 2\pi r(16 - r^2)$$

and then the volume is

$$\int_0^{4} \pi(32r - 2r^3)dr = \pi(16r^2 - r^4/2)|_0^4 = \pi(256 - 256/2) = 128\pi.$$

This is the first example of the "shell" technique, about which more later.

Note that I could *also* have done this problem by slicing along the x direction. So the cross section between x and dx has width dx, and looks pretty much like a parabola in the yz-plane, namely the parabola $z = y^2 + x^2$. And bounded below $z = 16$. So point out that you'd get

$$A(x) = \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 16 - x^2 - y^2 dx$$

which (I'm sparing you the steps) comes out to

$$A(x) = (16 - x^2)y - y^3/3|_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} = (4/3) * (16 - x^2)^{3/2}.$$

Now you'd have to integrate this over x! We can find the integral in our integral table (or attack it via trigonometric substitution) and frankly it is nasty. But it does give the right answer.