1 Introduction and Section 7.1: Shortcuts, Integral Tables, and Methods

1.1 Introduction to me and to the course

- **Before class:** Write name, Math 104: Calculus, office hour (Fine 808, M 2pm, Th 10am), ellenber@math.princeton.edu
- Introduce self: new Ph.D., specializing in number theory, Harvard, from Montgomery County, Maryland–we speak quickly–
- Hand out syllabus (simultaneous with above)
- I should be reached by e-mail, or in emergency by phone. I read e-mail all day, but not always at night–I'll inform you.
- Say, Adrian Banner will be your grader, but he's in Australia.
- Solicit questions on syllabus.
- Gospel of homework. You will only come to understand this material by doing. Even if you feel like you understand what I'm saying! There's a big difference between following a lecture when I'm here with you and facing a problem in that absolute existential solitude known as the final exam. **Do the homework!** Homework policy: Adrian will tell you when he gets here.
- Working together on homework: we encourage it. Talking things out is a good way to get over conceptual snags; usually it helps the explainer as much as, if not more than, the explainee; you come to understand something through explaining it, or, equally often, you realize you *don't* understand something when you try to explain it. However, try the problems on your own first–don't just divvy them up. Of course, write-ups must be done individually.
- Another good reason to do the homework is that they will count for a lot of your grade because *I do not give quizzes.*
- On lecture: I don't really love to lecture and I hope we'll be spending a sizable minority of our time in collaborative group work. Also: I hope to cover all material, but there will be times I have to skip. *You are responsible for the material in the assigned chapters.* Optimally, you will read the chapters *before* I lecture on them.

- Please do interrupt me. Inevitably I will use some concept or piece of language which seems clear to me but is not. If you are confused, so is someone else. Don't worry about slowing down the class–if in my opinion it's necessary to go on, we'll move on–let me worry about the pace.

- Solicit questions about the above.

- Pass out index cards. They should write their name, phone number, e-mail, (prospective) major, hometown, and draw a picture.

- Then turn to the person next to you. Solicit a piece of information about them which I will not otherwise find out in the course of this class. Introduce your partner.

- Announce no class on Monday.

1.2 Integration and differentiation: basic facts

Ask what people's background is–get people talking.

Just to start, let's discuss five true-false questions. Don't think too hard–I want your initial reactions.

1. If a car starts at a known point and travels for a minute, and we know the velocity of the car at every moment during the minute, we can calculate the final position of the car.

2. Same, with "velocity" replaced by "acceleration."

3. If a function f goes to infinity at 0, its derivative also goes to infinity at 0.

4. If a function f goes to infinity at 0, its indefinite integral also goes to infinity at 0.

5. If f is a function which is always positive,

$$\int_a^b f(x)dx > 0$$

   for all a, b.

Take a vote. Then ask them to rejoin their partner, discuss 3 minutes. Then take a new vote. Then solicit opinions on either side as usual. Concentrate especially on 2 and 5, which admit good answers of both "true" and "false."

1.3 First techniques of integration

Recall:

$$\int_a^b f(x)dx$$

signifies the area under the curve $y = f(x)$ from $x = a$ to $x = b$ (draw picture.)

And

$$\int f(x)dx = F$$

means that

$$\int_a^b f(x)dx = F(b) - F(a).$$

Solicit an example. Then point out, hey, funny, I did the same calculation and got 29 more, or whatever. This explains the constant of integration. The indefinite integral is the "area so far" function. Draw a picture.

The example is probably

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}.$$

Enjoy it, guys, this is the easiest one. The problem is this. We had no trouble calculating the derivative of a heinous function like

$$sin(x^3 - 4)e^x$$

because we have things like the Chain Rule and the product rule for computing derivatives. Integration is much more subtle and we do not have such nice formulas. But look on the bright side. First of all, functions like the above are sort of ad hoc and random anyway; maybe the fact is that it's easier to integrate functions that actually come up in practice. Moreover, we can look at things two ways. We could throw up our hands and say

$$\int_1^n \log(x)/x$$

is insoluble, and cry over it. (Aside: in this class, as in life, log is synonymous with ln.) Or we could just give this function a name, and call it $Li(n)$. After all, to say "We can't do an integral" is *not* to say the integral doesn't *exist*–it certainly exists! It is a function whose values we can calculate to any desired degree of accuracy! It just happens to be a function for which we don't yet have a name.

And it turns out that $Li(n)$ is almost exactly the number of prime numbers less than n! So this is actually a function worth having a name for! We shouldn't be prejudiced against it just because it is a so-called "elementary function." Indeed, let me expose the so-called "elementary functions" for what they are. What *is* this so-called function $\log x$, anyway?

Challenge: Define $\log x$ for me!

Then point out that it's an indefinite integral

$$\int_1^x (1/x)$$

So there you go.

So that's a lesson in why it's not so bad that we *can't* "do" certain integrals. However, our goal today is actually to do some integrals.

So here are the first techniques for solving

$$\int f(x)dx.$$

1. Write f as g'/g. For we know by the chain rule that

$$d(\log|g|)/dx = g'(x)/g(x)$$

(ask if notation is OK) (comment on dx–how we won't talk about what it is, but it's nice to keep track of it. How there should be "law of conservation of d"s and it will help you not get lost in substitutions, just as keeping track of units helps you not get lost in physics.)

Example.

$$\int 1dx$$

Write $1 = e^x/e^x$; then letting $g = e^x$ we get

$$\int e^x/e^x dx = \int g'(x)/g(x)dx = \log(g) = \log(|e^x|) = x + C.$$

Pretty good technique! How about another:

$$\int y^2/(y^3+5)dy = (1/3)\int 3y^2/(y^3+5) = (1/3)\log|y^3+5| + C.$$

And another:

$$\int \tan\theta d\theta = \int (\sin\theta/\cos\theta)d\theta = -\int(-\sin\theta)/\cos\theta d\theta = -\log|\cos\theta| + C.$$

This is a disguised version of the *substitution technique* we'll talk about on Wednesday.

2. The odd function trick.

$$\int_{-2}^{2} x^2 \sin x dx$$

Note that $f(x)$ is *odd*; that is, $f(-x) = -f(x)$. Draw the picture, argue that the integral is 0.

3. The circle trick.

$$\int_0^a \sqrt{a^2 - x^2} dx = (1/4)\pi a^2.$$

Draw the quarter-circle. What about

$$\int_0^{a/2} \sqrt{a^2 - 4x^2} dx?$$

Give them some time to think.

2 Tables

Pass out the tables. Try to get across that this was at times *very important*, and the numerical recipes we have today just weren't available.

So why do we still learn this today? Because sometimes you need to know more than the answer; sometimes you need to know "what's going on." Like, if you have some metal bar, you can use numerical methods to know how much weight you can put on it before it buckles. But how do you compute how this shifts with the weight of the bar? Numerical methods are inherently sort of inflexible.

Section 7.2: The substitution method

Recall the fundamental theorem of calculus:

$$F = \int f(x)dx$$

if and only if

$$dF/dx = f.$$

We can also write

$$dF(x) = f(x)dx$$

which allows us to restate the fundamental theorem of calculus in the following innocuous form:

$$F = \int dF$$

MORAL: Don't make too much distinction between "function" and "variable." Math doesn't care what letter you use–it can be x, F, a cow, whatever.

Ex: Compute

$$\int (3x^2 + 2)e^{x^3+2x}dx.$$

Idea: rewrite using $y = x^3 + 2x$. Then $dy = (3x^2 + 2)dx$, and we have

$$\int e^y dy = e^y + C = e^{x^3+2x} + C.$$

We can check by differentiating the above expression.

Ex:

$$\int dx = \int \frac{e^x}{e^x}dx = \int \frac{1}{F}dF = \log F + C = \log(e^x) + C = x + C.$$

Why so much work? Well, let me point out that if you *didn't* know the integral of $1/x$, this would be a *proof* of that fact! And if anyone ever asks you why the natural logarithm is "natural"–here is why.

Ex: Compute

$$\int x\sqrt{3x+1}dx.$$

We have to deal with that $\sqrt{3x+1}$ somehow. Solicit suggestions. Yes: substitute $u = 3x+1$, then note that $du = 3dx$ and $x = (u-1)/3$. Comment that we then compute $dx = (1/3)du$ in two different ways. We end up with

$$\int (1/3)(u-1)\cdot\sqrt{u}\cdot(1/3)du$$

$$= (1/9)\int u^{3/2} - u^{1/2}du = (1/9)((2/5)u^{5/2} - (2/3)u^{3/2}).$$

Then do the same with the substitution $v = \sqrt{3x+1}$. **Moral:** There is no one "correct" solution.

Talk about the definite integral; suggest that they write the boundaries out as $x = b$ and $x = a$, rather than just b and a, to avoid confusion.

Ex:(Problem 48, from p. 413 of Stein and Barcellos.)

$$\int_0^\pi \cos^2\theta d\theta.$$

Student 1 says, it is positive, because it's the integral of a function that's always positive.

Student 2 reasons,

$$\int_0^\pi \cos\theta\cos\theta$$

$$= \int_0^\pi \sqrt{1-sin^2\theta}cos\theta d\theta$$

$$= \int_0^0 \sqrt{1-u^2}du = 0.$$

Who is right? Take a vote, split into pairs, take another vote, discuss.

3 Section 7.3: Integration by Parts

First, announcements.

- I'm not teaching Wednesday–go to room 322,1201, or 110 as you like.

- Try to sell Forni and Swenton's classes.
- Hand out pages from Courant and Robbins on elementary functions.

First, discussion of the problem from last time. We can start by going ahead and giving the answer. Or rather, ask: when $u = 0$, what is $\sqrt{1-u^2}$? They will say 1 perhaps. Then argue that the answer is -1. Say, this shows you have to be careful with notation. Try to keep your common sense active. In this case, make your your inverse function is really an inverse function.

Today we talk about integration by parts. We can describe last time's lecture as "chain rule in reverse." We used the fact that

$$\frac{df(g)}{dx}(x) = \frac{df}{dx}(g(x))\frac{dg}{dx}(x)$$

to evaluate integrals that looked like the right hand side.

Integration by parts is "product rule in reverse."

As follows. Suppose I let $f(x) = x \log x$, and observe

$$df/dx = 1 \cdot \log x + x \cdot 1/x = \log x + 1.$$

It then follows by F T o' C that

$$\int (\log x + 1)dx = x \log x$$

(note we're experimenting with omitting the constant.)

But it gets better. Note that one part of this integral is particularly easy to do!

$$\int \log x dx + \int dx$$

$$= \int \log x dx + x = x \log x$$

and we have found a formula for the integral of $\log x dx$!

Let's try to go over what we did in terms of arbitrary functions. Write $u = \log x, v = x$. Then we had

$$d(uv)/dx = v(du/dx) + u(dv/dx)$$

or, more concisely

$$d(uv) = vdu + udv$$

so that

$$\int vdu + udv = uv$$

and

$$\int udv = uv - \int vdu$$

So the mantra is: take the guy you have, $f(x)dx$, write it as udv, in such a way that vdu is easy to calculate.

Ex: Compute

$$\int_{-infty}^{\infty} x^3 e^{-x^2} dx.$$

Now the question is: what are we going to do with that e^{-x^2}? Solicit suggestions. Conclude we need an x or a $2x$ or something. So let $u = x^2, dv = xe^{-x^2}dx$. So we compute $v = -(1/2)e^{-x^2}$. We are left with

$$\begin{aligned}\int x^3 e^{-x^2} dx &= -(1/2)x^2 e^{-x^2} - \int -2xdx(1/2)e^{-x^2} \\ &= (-1/2)x^2 e^{-x^2} - (1/2)e^{-x^2}.\end{aligned}$$

Evaluating from $-\infty$ to ∞, we get 0. Solicit opinions: could we have known that in advance? Repeat: *suppleness.* Just because we're in a different chapter doesn't mean we can't use what we already know!

Remark: when we have $x^a e^{-x^2}$ with a even, we don't have the odd function trick–might integration by parts help? Unfortunately, no!

And now, a remarkable example–we'll use the essentially *recursive* nature of integration by parts. That is–maybe vdu isn't an integral we can do–but at least maybe it's a little easier than what we had before–so we can just hit it with integration by parts once again!

Ex: Compute

$$\int \sin^5 xdx.$$

Well, set $u = \sin^4 x$ and $dv = \sin x dx$, so $v = -\cos x$ and we get

$$\begin{aligned}\int \sin^5 x dx &= -\sin^4 x \cos x - \int 4\sin^3 x \cos x(-\cos x)dx \\ &= -\sin^4 x \cos x - \int 4\sin^3 x(-1+\sin^2 x)dx \\ &= -\sin^4 x \cos x + 4\int \sin^3 x dx - 4\int \sin^5 x dx\end{aligned}$$

Now collect terms to get

$$5\int \sin^5 x dx = -\sin^4 x \cos x + 4\int \sin^3 dx$$

or

$$\int \sin^5 x dx = -(1/5)\sin^4 x \cos x + (4/5)\int \sin^3 dx$$

Now we still have an integral we don't know how to do–but you see that the method above will tell us in general

$$\int \sin^n x dx = -(1/n)\sin^{n-1} x \cos x + ((n-1)/n)\int \sin^{n-2} x dx$$

which means we can integrate $\sin^3$ in terms of sin, which we do understand.

Now let's prove something really really cool.

Note that if we make this a definite integral, we get

$$\int_0^{\pi/2} \sin^n x dx = (n-1)/n \int \pi/2_0 \sin^{n-2} x dx$$

Call this quantity I_n. (Following the notation of problem 39 in your book.) Now we see that the I_n's get very close to each other as we go on, since $(n-1)/n$ is very close to 1. But note that if n is *odd*, we get

$$I_n = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{4}{5}\frac{2}{3}\int \pi/2_0 \sin x dx$$

and this integral is 1, whereas we have

$$I_{n+1} = \frac{n}{n+1}\frac{n-2}{n}\cdots\frac{3}{4}\frac{1}{2}\int \pi/2_0 \sin^0 x dx$$

and we can replace the integral with $\pi/2$. But we know that as n goes to ∞, these guys get close together, so the quotient has to go to 1, which means

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots = \pi/2.$$

This is *Wallis's formula*, which appears in *Arithmetica Infinitorum*, published in 1656. Wallis is the guy who introduced the symbol ∞! And note that he preceded Newton–this is 15 years before Newton invents calculus. There was a cottage industry of calculating areas under curves, one by one. Without the F T o' C. Can you imagine?