Finite Flatness of Torsion Subschemes of Hilbert-Blumenthal Abelian Varieties

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Abstract

Let E be a totally real number field of degree d over $\mathbb{Q}$. We give a method for constructing a set of Hilbert modular cuspforms $f_1, \ldots, f_d$ with the following property. Let K be the fraction field of a complete dvr A, and let X/K be a Hilbert-Blumenthal abelian variety with multiplicative reduction and real multiplication by the ring of integers of E. Suppose n is an integer such that n divides the minimal valuation of $f_i(X)$ for all i. Then $X[n']/K$ extends to a finite flat group scheme over A, where n' is a divisor of n with n'/n bounded by a constant depending only on $f_1, \ldots, f_d$. When $E = \mathbb{Q}$, the theorem reduces to a well-known property of $f_1 = \Delta$. In the cases $E = \mathbb{Q}(\sqrt{2})$ and $E = \mathbb{Q}(\sqrt{5})$, we produce the desired pairs of Hilbert modular forms explicitly and show how they can be used to compute the group of Néron components of a Hilbert-Blumenthal abelian variety with real multiplication by E.

Introduction

Let A be a complete dvr with fraction field K, and valuation ord : $K^* \to \mathbb{Z}$. Let X/K be a Hilbert-Blumenthal abelian variety, or HBAV (Definition 1.3) with multiplicative reduction over K. Let n be an integer. We are interested in the following question:

> Does the torsion subscheme $X[n]/K$ extend to a finite flat group scheme over A?

Suppose X is an elliptic curve, with minimal discriminant Δ_{min}. It follows from the theory of Tate [22] that the above question has an affirmative answer whenever $\text{ord}(\Delta_{min})$ is divisible by n. In the present work, we will generalize this fact, replacing the modular form Δ with a finite set of Hilbert modular forms.

More precisely: we define a *discriminantal set of Hilbert modular forms* to be a set of modular forms whose q-expansions satisfy a certain convex-geometric condition (Definition 2.12.) In the following theorem, a totally real field E and a fractional ideal $\mathfrak{c} \subset E$ are fixed, and all HBAV's have real multiplication by the ring of integers of E.

Theorem (Corollary 2.16). *Let $f_1, \dots, f_r$ be a discriminantal set of $\mathfrak{c}$-Hilbert modular forms of level N (Definition 1.9.) Then there exists an integer $m_{\{f_i\}}$ such that the following is true. Suppose X/K is an HBAV with multiplicative reduction over A, and*

- *λ is a $\mathfrak{c}$-polarization for X; (Definition 1.5)*
- *ω is a Néron non-vanishing differential on X; (Definition 1.22)*
- *ι is a Néron N-level structure on X. (Definition 1.22)*

Let n be an integer such that $n | ord(f_i(X, \lambda, \omega, \iota))$ for all i, and let n' be the numerator of $n/m_{\{f_i\}}$ expressed in lowest terms.

Then $X[n']/K$ extends to a finite flat group scheme over A.

In fact, given the values of the $\text{ord}(f_i(X, \lambda, \omega, \iota))$, the valuation of the Tate parameter of X can be confined to a finite set of possibilities (Remark 2.17.) So the work here may be thought of as a small step in the direction of a "Tate's algorithm for HBAV's."

We also prove an existence theorem for discriminantal sets.

Theorem (Theorem 2.18). *Let E be a totally real number field of degree d over $\mathbb{Q}$, and $\mathfrak{c} \subset E$ a fractional ideal. Then there exists a discriminantal set of $\mathfrak{c}$-Hilbert modular forms for E, with cardinality d.*

Our ultimate aim is to investigate Diophantine equations associated to moduli spaces of HBAV's. The current work can be seen as the geometric portion of this arithmetic-geometric problem. To illustrate the ideas involved, consider the following situation. Suppose that $X/\mathbb{Q}$ is a modular elliptic curve, with semistable reduction over $\mathbb{Z}[1/m]$. Let $p \not| m$ be a prime, and let

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

be the Galois representation on the p-torsion points of X. Then the primes ℓ not dividing pm such that $X[p]/\mathbb{Q}_\ell$ extends to a finite flat group scheme over $\mathbb{Z}_\ell$ are exactly those which do not divide the Artin conductor of ρ. Moreover, the weight $k(\rho)$ is equal to 2 if and only if $X[p]$ extends to a finite flat group scheme over $\mathbb{Z}_p$. [20, Prop. 4].

It follows ([20, Prop. 7]) that an elliptic curve semistable over $\mathbb{Z}$ (and therefore modular) with ρ irreducible cannot have minimal discriminant a perfect pth power; for in that case $X[p]$ would extend to a finite flat group scheme over every $\mathbb{Z}_\ell$ (including $\mathbb{Z}_p$), so that ρ would have weight 2 and conductor 1, an impossibility by Ribet's theorem. One motivation for our work here is to extend arguments like the above to the general case of modular HBAV's, and to study Diophantine equations that stand in relation to Hilbert modular varieties as Fermat's equation does to the modular curve $X(2)$. In [3], we apply these ideas to show that solutions to the generalized Fermat equation $A^4 + B^2 = C^p$, under certain 2-adic conditions on A, B, C, would produce non-modular HBAV's. The HBAV's involved, however, are shown to be modular in [4]. This result and related ones will appear in a later paper.

The article is organized as follows. In §1, we review the necessary theory of Hilbert-Blumenthal abelian varieties and Hilbert modular forms. In §2.2, we define discriminantal sets and prove Corollary 2.16 above. In §2.3, we prove Theorem 2.18 above. Finally, in §2.4 we produce explicit discriminantal sets of modular forms for the cases $E = \mathbb{Q}[\sqrt{5}]$ and $E = \mathbb{Q}[\sqrt{2}]$.

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Notation

Throughout this article, we will fix

- E a totally real number field of degree d over $\mathbb{Q}$;
- $\rho_1, \ldots, \rho_d$ the embeddings of E into $\mathbb{R}$;
- $\mathcal{O}$ the ring of integers of E;
- $\mathfrak{o}$ the unit ideal in $\mathcal{O}$;

- $\mathfrak{d}$ the different of $\mathcal{O}/\mathbb{Z}$;
- U the group of units of $\mathcal{O}$;
- U^{++} the group of totally positive units of $\mathcal{O}$;
- A a complete dvr;
- K the fraction field of A;
- $\bar{K}$ an algebraic closure of K;
- "ord" the valuation homomorphism $K^* \to \mathbb{Z}$;

If $\mathfrak{c}$ is a fractional ideal in $\mathcal{O}$, we denote by $\mathfrak{c}^{++}$ the set of totally positive elements of $\mathfrak{c}$. More generally, if $\mathfrak{m}$ is any projective rank 1 $\mathcal{O}$-module, we define a *positivity structure* on $\mathfrak{m}$ to be a subset $\mathfrak{m}^{++}$ which is closed under addition and multiplication by $\mathfrak{o}^{++}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are projective rank 1 $\mathcal{O}$-modules, we mean by $\mathfrak{ab}$ the projective rank 1 $\mathcal{O}$-module $\mathfrak{a} \otimes_{\mathcal{O}} \mathfrak{b}$.

The superscript $^\vee$ signifies the dual abelian variety when applied to an abelian variety, and the $\mathbb{Z}$-dual when applied to an $\mathcal{O}$-module.

If X/S is any scheme, and $s = \operatorname{Spec} k$ is a point of S, we will often write X_k to denote the fiber product $X \times_S s$.

If X/S is an group scheme, we denote the identity section by

$$e_X : S \to X.$$

If $\nu : X \to X$ is an endomorphism of X, we denote the kernel of ν by $X[\nu]$. In particular, $X[n]$ is the subgroup scheme of n-torsion points of X.

If X/S is a smooth group scheme, we denote the locally free $\mathcal{O}_S$-module $e_X^*\Omega^1_{X/S}$ by $\underline{\omega}(X/S)$. The dual of $\underline{\omega}(X/S)$ (as $\mathcal{O}_S$-module) will be called $\operatorname{Lie}(X/S)$.

1 HBAV's and Hilbert modular forms

1.1 Elliptic curves with multiplicative reduction

The goal of this paper is to generalize a rather innocuous fact about elliptic curves with multiplicative reduction over non-archimedean local fields.

Let X/K be an elliptic curve with split multiplicative reduction over A. We know from the theory of Tate [22] that X can be thought of as a "quotient" of a torus by a discrete group:

$$X \cong \mathbb{G}_m/q_X^{\mathbb{Z}}$$

for some parameter $q_X \in K^*$ with $\operatorname{ord}(q_X) > 0$. For our purposes, we will not need to delve too closely into the meaning of the isomorphism written above.

Suffice it to say that there exists an elliptic curve over $\mathbb{Z}((q))$, called the *Tate curve*, or Tate(q), fitting into a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Tate}(q) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, K & \xrightarrow{\phi_X} & \mathrm{Spec}\, \mathbb{Z}((q)) \end{array}$$

where ϕ_X is induced by the ring homomorphism sending q to q_X.

In view of the above, we are naturally interested in computing the parameter q_X. Indeed, the integer $\mathrm{ord}(q_X)$ is already an important invariant of X; for instance, the component group of the special fiber of the Néron model of X is a cyclic group of order $\mathrm{ord}(q_X)$.

Theorem 1.1. *Let $\Delta_{min}(X)$ be the minimal discriminant of X. Then*

$$ord(\Delta_{min}(X)) = ord(q_X).$$

Proof. The minimal discriminant of Tate(q) is the usual q-expansion

$$q - 24q^2 + \dots$$

Thus, the minimal discriminant of X is

$$q_X - 24q_X^2 + \dots$$

and the theorem is immediate. □

Corollary 1.2. *If n is a rational integer dividing $ord(\Delta_{min}(X))$, the group scheme $X[n]/K$ extends to a finite flat group scheme over A. In particular, let $\mathcal{X}/A$ be the Néron model of X. Then the group scheme $\mathcal{X}[n]$ is finite and flat over A.*

Proof. Since multiplication by n is flat, only finiteness needs to be proved.

Let $\mathcal{X}^0$ be the identity component of $\mathcal{X}$; let $\mathcal{X}_s$ be the fiber of $\mathcal{X}$ over the closed point of Spec A, and let $\mathcal{X}_s^0$ be the identity component of $\mathcal{X}_s$. Then $\mathcal{X}_s^0[n]$ is a finite flat group scheme over s of degree n.

Let A' be an unramified extension of A such that the residue field k' of A' is separably closed. Fix an embedding of k' into an algebraic closure $\bar{k}$ of k.

It follows from Tate's algorithm that

$$\Phi := \mathcal{X}(A')/\mathcal{X}^0(A') \cong \mathbb{Z}/mZ$$

where $m = \mathrm{ord}(q_X)$ is a multiple of n. In particular, $\Phi[n]$ has order n.

Let $x_\alpha \in \mathcal{X}(A')$ be a point representing an element α of $\Phi[n]$. Then $nx_\alpha \in \mathcal{X}^0(A')$. Let $\bar{x}_\alpha$ be the restriction of x_α to the geometric fiber $\mathcal{X}_{\bar{s}} = \mathcal{X}_s \times_k \bar{k}$.

Multiplication by n is surjective on $\bar{k}$-points of $\mathcal{X}_{\bar{s}}^0$; thus, we may choose $y \in \mathcal{X}_{\bar{s}}^0(\bar{k})$ such that $ny = n\bar{x}_\alpha$.

Now let $x'_\alpha = \bar{x}_\alpha - y$. Then $x'_\alpha + \mathcal{X}^0_{\bar{s}}[n]$ is a subscheme of $\mathcal{X}_{\bar{s}}[n]$, of degree n over $\bar{s}$. Moreover, $x'_\alpha + \mathcal{X}^0_{\bar{s}}[n]$ and $x'_\beta + \mathcal{X}^0_{\bar{s}}[n]$ are disjoint for $\alpha \neq \beta$, since they lie in distinct components of $\mathcal{X}_{\bar{s}}$. We conclude that $\mathcal{X}_{\bar{s}}[n]$ is a group scheme of degree at least n^2 over $\bar{s}$, whence $\mathcal{X}_s[n]$ is a group scheme of order at least n^2 over s.

Now $\mathcal{X}[n]$ is a quasi-finite scheme over a complete dvr, and as such it can be written as the disjoint union of a finite scheme Y' and a scheme Y'' which is void over the special point [6, (6.2.6)]. Since $\mathcal{X}[n]$ is flat, Y' has equal degree over the generic and special points. But we have shown that the degree of $\mathcal{X}[n]$ over the special point at least n^2, which is the degree of $\mathcal{X}[n]$ over the generic point. Therefore, Y'' is void, and $\mathcal{X}[n]$ is finite, as desired. □

Our aim is to generalize the above corollary to the case where X is a Hilbert-Blumenthal abelian variety; that is, we want to give a criterion, in terms of the values of certain modular forms evaluated at X, for $X[n]/K$ to extend to a finite flat group scheme over A.

1.2 Definitions and examples

The basic theory of Hilbert-Blumenthal abelian varieties (HBAV's), also called *abelian varieties with real multiplication*, can only be summarized in this small space. For more detail and justification, we refer the reader to Katz [10, ch. 1], whose definitions and notation we follow closely in this paper. Another excellent reference is [23].

Definition 1.3. A *Hilbert-Blumenthal abelian variety* (HBAV) over a ring R is a pair (X, ϕ), where

- X is an abelian scheme of dimension d over $\operatorname{Spec} R$;
- $\phi : \mathcal{O} \hookrightarrow \operatorname{End}(X)$ is an injection making $\operatorname{Lie}(X/R)$ a locally (on R) free rank-1 $\mathcal{O} \otimes_{\mathbb{Z}} R$ module.

We will refer to X as "an HBAV with real multiplication by $\mathcal{O}$" if there is any ambiguity about what totally real ring of integers is acting. We will denote the endomorphism $\phi(\alpha)$ by $[\alpha]$. When no confusion is likely, an HBAV (X, ϕ) will be referred to simply as X.

Remarks 1.4.

1. If X/K is an abelian scheme of dimension d over a field of characteristic 0, endowed with an injection $\phi : \mathcal{O} \hookrightarrow \operatorname{End}(X)$, the condition on $\operatorname{Lie}(X/K)$ is automatically satisfied [16, Prop. 1.4].
2. An elliptic curve is an HBAV with real multiplication by $\mathbb{Z}$.
3. Brumer has shown [2] that the Jacobian of the genus-2 curve

$$\begin{aligned} C : y^2 &= x^6 + 2cx^5 + (c^2 + 2c + 2 - bd)x^4 \\ &+ (2c^2 + 2c + 2 + b - 2bd - 4d)x^3 \\ &+ (c^2 + 4c + 5 + 3b - bd)x^2 + (2c + 6 + 3b)x + (b + 1) \end{aligned} \tag{1.2.1}$$

is an HBAV over $\mathbb{Q}(b,c,d)$ with real multiplication by $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

4. The endomorphism $[\alpha]$ is an isogeny [1, 7.3,Def. 4] for all $\alpha = 0$, since it factors through $[n]$ for some rational integer n, and $[n]$ is an isogeny.

Following Katz [10], we begin by defining some extra structures on HBAV's.

Definition 1.5. Let X be an HBAV and $\mathfrak{c}$ a fractional ideal of $\mathcal{O}$. Let $X \otimes_{\mathcal{O}} \mathfrak{c}$ be the HBAV representing the functor $T \mapsto X(T) \otimes_{\mathcal{O}} \mathfrak{c}$, and let

$$\epsilon : \mathfrak{c} \to \mathrm{Hom}_{\mathcal{O}}(X, X \otimes_{\mathcal{O}} \mathfrak{c})$$

be the evident homomorphism. Let $P(X)$ be the $\mathcal{O}$-module of $\mathcal{O}$-linear symmetric morphisms from X to $X^{\vee}$. We say an element of $P(X)$ is positive if it is a polarization of X as an abelian scheme.

Then a $\mathfrak{c}$-*polarization* of X (as an HBAV) is an isomorphism

$$\lambda : X \otimes_{\mathcal{O}} \mathfrak{c} \to X^{\vee}$$

such that the map $\mathfrak{c} \to P(X)$ given by

$$c \to \lambda \circ \epsilon(c)$$

is an isomorphism which associates the totally positive elements of $\mathfrak{c}$ to the totally positive elements of $P(X)$.

Remark 1.6. If $\mathfrak{c}'$ is a fractional ideal in the same narrow ideal class as $\mathfrak{c}$, then any $\mathfrak{c}$-polarization of an HBAV X naturally gives rise to a $\mathfrak{c}'$-polarization. So the choice of $\mathfrak{c}$ is essentially only a choice of a narrow ideal class of $\mathcal{O}$.

Since $\underline{\omega}_{X/R}$ is R-dual to Lie(X/R), it is locally isomorphic to $\mathfrak{d}^{-1} \otimes R$, and is hence also a locally (on R) free $\mathcal{O} \otimes R$-module.

Definition 1.7. A *non-vanishing differential* on a HBAV X is a generator for $\underline{\omega}_{X/R}$ as $\mathcal{O} \otimes R$-module.

Equivalently, we may say that a non-vanishing differential is an isomorphism between Lie(X/R) and $\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} R$.

Finally, we stipulate a notion of level structure.

Definition 1.8. An N*-level structure* (or $\Gamma_{00}(N)$*-structure*) for a HBAV X is an $\mathcal{O}$-linear closed immersion of group schemes over R

$$\iota : \mathfrak{d}^{-1} \otimes \mu_N \hookrightarrow X,$$

where μ_N/R is the (finite) group scheme of N-th roots of unity.

In case $N = 1$, we write ι_1 for the trivial level structure.

Now we are ready to define Hilbert modular forms.

Definition 1.9. Suppose $d > 1$. A $\mathfrak{c}$-*Hilbert modular form of level N and weight k over R_0* is a function on quadruples $(X, \lambda, \omega, \iota)$, where

- X is an HBAV over an R_0-algebra R;
- λ is a $\mathfrak{c}$-polarization of X;
- ω is a non-vanishing differential on X;
- ι is an N-level structure for X,

such that

- $f(X/R, \lambda, \omega, \iota)$ is an element of R depending only on the R-isomorphism class of $(X, \lambda, \omega, \iota)$;
- f commutes with any extension of scalars $R \to R'$;
- For any x in $(\mathcal{O} \otimes R)^*$,

$$f(X, \lambda, x\omega, \iota) = \mathbf{N}_{\mathbb{Q}}^{E}(x)^{-k} f(X, \lambda, \omega, \iota).$$

The R_0-module of such functions is denoted by $M_{\mathcal{O}}^k(\mathfrak{c}, N; R_0)$. The ring of modular forms of arbitrary integral weight is called $M_{\mathcal{O}}^*(\mathfrak{c}, N; R_0)$.

Remark 1.10. If $d = 1$ (that is, if $\mathcal{O} = \mathbb{Z}$) another condition on f is necessary, namely that it be "holomorphic at the cusps." That this condition is unnecessary when $d > 1$ is the content of the Koecher principle (Proposition 1.17 below.)

Example 1.11. Suppose $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and let $\mathfrak{c} = \mathfrak{d}$. Since the narrow class group of $\mathcal{O}$ is trivial, this is essentially the only choice (see Remark 1.6); we choose $\mathfrak{d}$ rather than $\mathfrak{o}$ for notational convenience later on.

Then

$$M_{\mathcal{O}}^*(\mathfrak{d}, 1; \mathbb{C}) \cong \mathbb{C}[\phi_2, \chi_5, \chi_6, \chi_{15}]/R$$

where the generators are modular forms of weights $2, 5, 6, 15$, and R is a relation in weight 30. (See Hirzebruch [8]). Nagaoka [15] proved that the forms χ_6 and $\chi_{12} := (1/4)(\chi_6^2 - \phi_2\chi_5)$ are in fact base changes of forms in $M_{\mathcal{O}}^*(\mathfrak{d}, 1; \mathbb{Z})$.

1.3 Semi-HBAV's

We will often want to consider a class of group schemes somewhat more general than HBAV's. Recall that a semi-abelian scheme X over a base S is a smooth commutative group scheme over S, each of whose fibers is an extension of an abelian variety by a torus. If the fiber over $s \in S$ is an extension of an abelian variety by a split torus, we say X is split at s.

Definition 1.12. A *semi-HBAV* over $\operatorname{Spec} R$ is a pair (X, ϕ), where

- X is a semi-abelian scheme of dimension d over R;
- $\phi : \mathcal{O} \hookrightarrow \operatorname{End}(X)$ is a homomorphism making $\operatorname{Lie}(X/R)$ a locally (on R) free rank-1 $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-module.

Note that it follows from the second condition in Definition 1.12 that each fiber of a semi-HBAV is either an abelian variety or a torus. We say a semi-HBAV is *split* if each of its geometric fibers is either an abelian variety or a split torus.

If K is the fraction field of a complete dvr A, and Y/K is an abelian variety, we say that Y has multiplicative reduction (resp. split multiplicative reduction) if the identity component of its Néron model is a semi-abelian variety whose special fiber is a torus (resp. semi-abelian variety whose special fiber is a split torus.)

By analogy with the HBAV case, we define a non-vanishing differential on a semi-HBAV X/R to be a generator for the $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-module $\underline{\omega}_{X/R}$, and an N-level structure to be an $\mathcal{O}$-linear closed immersion

$$\iota : \mathfrak{d}^{-1} \otimes \mu_N \hookrightarrow X.$$

We have already seen that elliptic curves with split multiplicative reduction over A can be expressed as pullbacks of a Tate curve over $\mathbb{Z}[[q]]$. The situation for HBAV's with split multiplicative reduction (indeed, for general abelian varieties with split multiplicative reduction) is very much the same; all HBAV's with split multiplicative reduction will be pullbacks of a certain "Tate HBAV."

1.4 The Tate HBAV and q-expansion

In order to define the Tate HBAV, we must first describe the power series ring over which it is defined. To this end we make the following definitions.

Let $\mathfrak{m}$ be a projective rank 1 $\mathcal{O}$-module endowed with a positivity structure $\mathfrak{m}^{++}$.

Definition 1.13. A linear form $L : \mathfrak{m} \to \mathbb{Z}$ is *positive* if it sends all positive elements of $\mathfrak{m}$ to positive integers.

Definition 1.14. Let S be a set of d linearly independent positive linear forms. An element $m \in \mathfrak{m}$ is *S-semipositive* if $L(m) \geq 0$ for every $L \in S$. The set of S-semipositive elements of $\mathfrak{m}$ is denoted $\mathfrak{m}^S$.

Let $\mathfrak{a}$ be a projective rank 1 $\mathcal{O}$-module, $\mathfrak{c}$ a fractional ideal of $\mathcal{O}$, and set $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. We define a positivity structure on $\mathfrak{a}\mathfrak{b}$ by defining $\mathfrak{a}\mathfrak{b}^{++}$ to be the semigroup generated by all elements of the form $a \otimes ac^{-1}$, with $a \in \mathfrak{a}$ and $c \in \mathfrak{c}^{++}$.

Now define

$$\mathbb{Z}[\mathfrak{ab}, S] = \mathbb{Z}[\dots, q^{\alpha}, \dots]/(q^0 - 1, q^{\alpha_1+\alpha_2} - q^{\alpha_1}q^{\alpha_2})$$

where $\alpha, \alpha_1, \alpha_2$ vary over $(\mathfrak{ab})^S$. Let I_S be the ideal generated by q^{α} for all $\alpha \in (\mathfrak{ab})^S - \{0\}$, and let $\mathbb{Z}[[\mathfrak{ab}, S]]$ be the completion of $\mathbb{Z}[\mathfrak{ab}, S]$ in the I_S-adic topology. Let I be the ideal in $\mathbb{Z}[[\mathfrak{ab}, S]]$ generated by q^{α} for all *totally positive* α. We denote by $\mathbb{Z}((\mathfrak{ab}, S))$ the localization of $\mathbb{Z}[[\mathfrak{ab}, S]]$ away from I.

Now $\mathbb{Z}[[\mathfrak{ab}, S]]$ is a normal Noetherian domain complete with respect to I. Moreover, the pairing

$$q : \mathfrak{b} \otimes \mathfrak{a} \to \mathbb{G}_m(\mathbb{Z}((\mathfrak{ab}, S)))$$

defined by

$$q(\beta \otimes \alpha) = q^{\beta\alpha}.$$

is a polarized set of periods in the sense of the construction of Mumford [14]. Therefore, there exists an abelian variety $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ over $\mathbb{Z}((\mathfrak{ab}, S))$ which we may think of as the quotient

$$\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m/q_{\mathfrak{a}}(\mathfrak{b}),$$

where

$$q_{\mathfrak{a}} : \mathfrak{b} \to \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m(\mathbb{Z}((\mathfrak{ab}, S)))$$

is the homomorphism induced by q.

We record below some basic facts about the Tate HBAV, which can readily be derived from Mumford's work in [14]. (See [3] for more details.)

1. $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is an HBAV, which extends to a split semi-HBAV $\widetilde{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q)/\mathbb{Z}[[\mathfrak{ab}; S]]$.

2. $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$-polarization λ_{can}.

3. There is a canonical isomorphism

$$\omega_{\mathfrak{a}} : \mathrm{Lie}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)) \to \mathrm{Lie}(\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m/\mathbb{Z}((\mathfrak{ab}, S)))$$

   which extends to an isomorphism between $\mathrm{Lie}(\widetilde{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q))$ and $\mathrm{Lie}(\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m/\mathbb{Z}[[\mathfrak{ab}, S]])$.

   In particular, if R is a ring, any isomorphism

$$j : \mathfrak{a} \otimes_{\mathbb{Z}} R \xrightarrow{\sim} \mathfrak{o} \otimes_{\mathbb{Z}} R$$

   induces a non-vanishing differential on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \times_{\mathbb{Z}} R$ by composition with $\omega_{\mathfrak{a}}$. We denote this non-vanishing differential by $\omega(j)$.

4. Let N be a rational integer. There exists an exact sequence of group schemes over $\mathbb{Z}((\mathfrak{ab}, S))$

$$0 \to \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes \mu_N \xrightarrow{\iota_{\mathfrak{a}}} \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)[N] \to ((1/N)\mathbb{Z}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathfrak{b} \to 0,$$

and $\iota_{\mathfrak{a}}$ extends to a morphism of group schemes over $\mathbb{Z}[[\mathfrak{ab}, S]]$:

$$\iota_{\mathfrak{a}} : \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes \mu_N \to \widetilde{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q)[N].$$

So any isomorphism

$$\epsilon : \mathfrak{o}/N\mathfrak{o} \xrightarrow{\sim} \mathfrak{a}/N\mathfrak{a}$$

induces a level N structure on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by composition with $\iota_{\mathfrak{a}}$. We denote this level N structure by $\iota(\epsilon)$. Note that $\iota(\epsilon)$ extends to a level N structure on $\widetilde{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q)$.

Suppose f is a $\mathfrak{c}$-Hilbert modular form of level N over a ring R_0, and $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is a quadruple where

- $\mathfrak{a}$ is a projective rank 1 $\mathcal{O}$-module and $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$;
- $j : \mathfrak{a} \otimes_{\mathbb{Z}} R \xrightarrow{\sim} \mathfrak{o} \otimes_{\mathbb{Z}} R$ is an isomorphism, for some R_0-algebra R;
- $\epsilon : \mathfrak{o}/N\mathfrak{o} \xrightarrow{\sim} \mathfrak{a}/N\mathfrak{a}$ is an isomorphism.

Such a quadruple is called a *cusp* (or a *cusp of level* N) over R.

In case $N = 1$, we denote by ϵ_1 the canonical isomorphism from $\mathfrak{o}/\mathfrak{o}$ to $\mathfrak{a}/\mathfrak{a}$.

Definition 1.15. The *q-expansion* of f at the cusp $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is the value

$$f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \times_{\mathbb{Z}} R, \lambda_{can}, \omega(j), \iota(\epsilon)).$$

A priori, a q-expansion of a modular form at a cusp over R is an element of $R_0((\mathfrak{ab}, S)) \otimes_{R_0} R$. In fact, q-expansions are constrained to lie in a smaller ring. We record three standard facts below.

Proposition 1.16. *The q-expansion of f at $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is independent of the choice of S.*

Proof. Let S_1, S_2 be two choices of S. Suppose

$$(\mathfrak{ab})^{S_2} \subset (\mathfrak{ab})^{S_1}.$$

Then the Tate HBAV computed using S_1 is just a pullback of the Tate HBAV computed using S_2, via the natural inclusion $\mathbb{Z}[[\mathfrak{ab}; S_2]] \subset \mathbb{Z}[[\mathfrak{ab}; S_1]]$. Therefore, the q-expansions computed using S_1 are the same as those using S_2.

The general result follows from choosing S_3 such that

$$(\mathfrak{ab})^{S_3} \subset (\mathfrak{ab})^{S_i} (i = 1, 2).$$

□

Proposition 1.17 (Koecher principle). *Suppose $d > 1$. Let f be a modular form whose q-expansion at a cusp $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is*

$$\sum_{\alpha \in \mathfrak{ab}} a_\alpha(f) q^\alpha.$$

Then $a_\alpha = 0$ unless $\alpha = 0$ or $\alpha \in \mathfrak{ab}^{++}$.

Proof. See [16, Prop. 4.9]. □

Proposition 1.18. *Let f be a modular form whose q-expansion at $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is*

$$\sum_{\alpha \in \mathfrak{ab}} a_\alpha(f) q^\alpha,$$

and let $v \in U$ be congruent to 1 (mod N). Then $a_{v^2\alpha}(f) = (\mathbf{N}_{\mathbb{Q}}^{E} v)^k a_\alpha(f)$ for all $\alpha \in \mathfrak{ab}$.

Proof. Consider the base change of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ induced by the automorphism of $\mathbb{Z}[[\mathfrak{ab}; S_1]]$ sending q to q^u. The result then follows from invariance of Hilbert modular forms under base change. □

1.5 Uniformization of HBAV's

As described above, HBAV's with split multiplicative reduction are pullbacks of the Tate HBAV, just as is the case for elliptic curves.

We say a homomorphism $q : \mathfrak{ab} \to \mathbb{G}_m(K)$ is *positive* if it takes $\mathfrak{ab}^{++}$ to A. Any such q yields a homomorphism

$$q_{\mathfrak{a}} : \mathfrak{b} \to \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m(K)$$

and, for suitably chosen S, a map of schemes

$$\tilde{q} : \mathrm{Spec}\, A \to \mathrm{Spec}\, \mathbb{Z}[[\mathfrak{ab}, S]].$$

Theorem 1.19 (Uniformization Theorem). *Let X/K be a $\mathfrak{c}$-polarized H-BAV with split multiplicative reduction over A. Then there exists a projective rank 1 $\mathcal{O}$-module $\mathfrak{a}$, a set of d linearly independent positive linear forms S, and a positive homomorphism*

$$q_X : \mathfrak{ab} \to \mathbb{G}_m(K)$$

(where $\mathfrak{b} = \mathfrak{ac}^{-1}$) such that X/K is isomorphic (as a $\mathfrak{c}$-polarized HBAV) with the pullback of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by $\tilde{q}_X$. The choice of q_X is unique up to multiplication by squares of units of $\mathcal{O}$.

This theorem follows from a more general theorem on abelian varieties with split purely multiplicative reduction, proved independently by Mumford [14] and Raynaud [18]. A thorough treatment of this theorem, and still more general ones, can be found in the book of Faltings and Chai [5].

Remark 1.20. If X is isomorphic to a pullback of a Tate HBAV $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by $\tilde{q}_X$, as in the theorem, we will write

$$X \cong (\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{X;\mathfrak{a}}(\mathfrak{b}).$$

Corollary 1.21. *Let X/K be an HBAV with multiplicative reduction, and let $\mathcal{X}^0/A$ be the identity component of the Néron model of X. Then $\mathcal{X}^0$ is a semi-HBAV.*

Proof. After passing to an étale extension, we may assume that X has split multiplicative reduction. Then X is the pullback of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by some $\tilde{q}_X$. The pullback of $\widetilde{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q)$ by $\tilde{q}_X$ is then a semi-HBAV over A whose generic fiber is X; such a semi-abelian variety is isomorphic to $\mathcal{X}^0$ by [1, 7.4, Prop. 3]. □

Definition 1.22. Let $X, \mathcal{X}^0$ be as above. A *Néron differential* on X is a non-vanishing differential which is the restriction to K of a non-vanishing differential on $\mathcal{X}^0/A$. Likewise, a *Néron N-level structure* on X is an N-level structure which is the restriction to K of an N-level structure on $\mathcal{X}^0/A$.

Proposition 1.23. *Suppose X/K is an HBAV such that*

$$X \cong (\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{X;\mathfrak{a}}(\mathfrak{b}).$$

Suppose ω is a Néron non-vanishing differential on X and ι is a Néron N-level structure on X. Then there exists a cusp $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ over A such that $\omega = \tilde{q}_X^* \omega(j)$ and $\iota = \tilde{q}_X^* \iota(\epsilon)$.

Proof. The formal completion of $\mathcal{X}^0$ over $\mathrm{Spf}(A)$ is isomorphic to that of $\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m$ [14, §3]. So the N-level structures

$$\iota : \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow \mathcal{X}^0$$

are in bijection with maps

$$\iota : \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m$$

(by [7, 5.4.1]), which are in turn classified by isomorphisms

$$\epsilon : \mathfrak{o}/N\mathfrak{o} \xrightarrow{\sim} \mathfrak{a}/N\mathfrak{a}.$$

Likewise, [7, 5.1.6] yields an isomorphism

$$\mathrm{Lie}(\mathcal{X}^0/A) \cong \mathrm{Lie}(\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m/A) = \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} A$$

which associates to every $\omega : \mathrm{Lie}(\mathcal{X}^0/A) \cong \mathfrak{d}^{-1} \otimes_Z A$ an isomorphism $j : \mathfrak{a} \otimes_{\mathbb{Z}} A \cong \mathfrak{o} \otimes_{\mathbb{Z}} A$ such that $\omega = \tilde{q}_X^* \omega(j)$. □

The power of the Uniformization Theorem is that it allows us to reduce many questions about the geometry of HBAV's to questions about the parameter q_X. In particular, we will see in Proposition 1.25 below that the question of whether the n-torsion subscheme of X can be extended to a finite flat group scheme over A can sometimes be answered immediately via knowledge of q_X.

Definition 1.24. Let

$$q : \mathfrak{ab} \to K^*$$

be a positive homomorphism. Then $\mathrm{ord}(q)$ is the element $x \in (\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1})^{++}$ such that

$$\mathrm{ord}(q(y)) = \mathrm{Tr}(y \cdot \mathrm{ord}(q))$$

for all y in $\mathfrak{ab}$.

Note that if X/K is a $\mathfrak{c}$-polarized HBAV with split multiplicative reduction over A, the quantity $\mathrm{ord}(q_X)$ is defined up to multiplication by squares of units.

For the following proposition, note that, by [1, 7.3.6], the isogeny $\nu : X \to X$ extends uniquely to an isogeny $\nu : \mathcal{X} \to \mathcal{X}$. In particular, $\nu : \mathcal{X} \to \mathcal{X}$ is flat.

Proposition 1.25. *Let X/K be a $\mathfrak{c}$-polarized HBAV with multiplicative reduction over A. Let A' be an etale extension of A, with fraction field K', such that $X' = X \times_K K'$ has split multiplicative reduction. Write X' as*

$$X' \cong (\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{X';\mathfrak{a}}(\mathfrak{b}),$$

and let ν be an element of $\mathcal{O}$ such that

$$\mathrm{o}rd(q_{X'}) \in \nu\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}.$$

Then $X[\nu]/K$ extends to a finite flat group scheme over A. In particular, let $\mathcal{X}/A$ be the Néron model of X. Then the group scheme $\mathcal{X}[\nu]$ is finite and flat over A.

Proof. We follow closely the proof of Corollary 1.2.

Since ν is flat, only finiteness needs to be proved.

Let $\mathcal{X}'/A'$ be the Néron model of X'. Since formation of Néron model commutes with etale base change, $\mathcal{X}'[\nu]$ is isomorphic to $\mathcal{X}[\nu] \times_A A'$. Finiteness satisfies fpqc descent; thus, if $\mathcal{X}'[\nu]/A'$ is finite, so is $\mathcal{X}[\nu]/A$. So it suffices to consider the case $A = A'$. Moreover, we may suppose the residue field k of A to be separably closed.

Note that, by [12, 12.12], the generic fiber of $\mathcal{X}[\nu]$ has degree $\mathbf{N}(\nu)^2$ over $\mathrm{Spec}\, K$.

Let s be the closed point of $\mathrm{Spec}\, A$, and let $\mathcal{X}_s, \mathcal{X}_s^0$ be the special fibers of $\mathcal{X}, \mathcal{X}^0$. Recall that $\nu : \mathcal{X}^0 \to \mathcal{X}^0$ was produced in [14] as the algebrization of a certain formal morphism; from that description, it follows that

$$\mathcal{X}_s^0[\nu] \cong (\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)[\nu],$$

and in particular $\mathcal{X}_s^0[\nu]$ is a finite group scheme of degree $\mathbf{N}\nu$ over s.

It follows from [5, III.8.2] that

$$\Phi := \mathcal{X}_s/\mathcal{X}_s^0 = \mathfrak{d}^{-1}\mathfrak{a}^{-1}/\mathrm{ord}(q_X)\mathfrak{b} \tag{1.5.2}$$

By the functoriality of Mumford's construction, (1.5.2) is an equality, not only of groups, but of $\mathcal{O}$-modules. In particular, $\Phi[\nu]$ has order $\mathbf{N}\nu$.

The finiteness of $\mathcal{X}[\nu]$ now follows exactly as in Corollary 1.2. □

2 Discriminantal sets of Hilbert modular forms

The main part of this paper has as its goal to derive information about $\mathrm{ord}(q_X)$ from the values of Hilbert modular forms, and thereby, via Proposition 1.25, to obtain criteria guaranteeing that a torsion subscheme $X[n]/K$ extends to a finite flat group scheme over A.

2.1 Vertices and edges

In this section we collect some elementary results about the convex geometry of subsets of projective rank 1 $\mathcal{O}$-modules.

To begin with, let $\Lambda \subset \mathbb{R}^d$ be the image of an embedding $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$; that is, Λ is a lattice in $\mathbb{R}^d$. We denote by $(\mathbb{R}^d)^{++}$ the subset of $\mathbb{R}^d$ consisting of points with all coordinates positive, and by Λ^{++} the intersection of Λ with $(\mathbb{R}^d)^{++}$.

For the following definitions, let S be a non-empty subset of Λ^{++}.

Definition 2.1. An element $\alpha \in S$ is a *vertex* of S if there exists a linear form $L : \Lambda \to \mathbb{R}$ such that

$$L(\alpha) < L(\alpha'), \forall \alpha' \in S - \{\alpha\} \tag{2.1.3}$$

A form L satisfying (2.1.3) is said to be *minimized* at α.

Definition 2.2. A pair $\{\alpha, \beta\}$ of vertices of S is an *edge* if there exists a linear form $L : \Lambda \to \mathbb{R}$ such that

$$L(\alpha) = L(\beta)$$

and

$$L(\alpha) \leq L(\alpha'), \forall \alpha' \in S.$$

In this case, we say L is minimized at $\{\alpha, \beta\}$.

A linear form $L : \Lambda \to \mathbb{R}$ is said to be *positive* if it maps Λ^{++} to $\mathbb{R}^+$.

Proposition 2.3. *Every positive linear form L is minimized at either an edge or a vertex.*

Proof. Let α_0 be an element of S. Define a subset R of $(\mathbb{R}^d)^{++}$ by

$$R = \{x \in (\mathbb{R}^d)^{++} : L(x) \leq L(\alpha_0)\}.$$

Let

$$y_i = L(0, \ldots, 1, \ldots, 0),$$

where the 1 above is in the ith coordinate. Then R lies inside the region

$$\{x = (x_1, \ldots, x_d) : 0 \leq x_i \leq y_i^{-1} L(\alpha_0)\}.$$

So R is bounded, and since $S \subset \Lambda$ is discrete, the set

$$S \cap R = \{\alpha \in S : L(\alpha) \leq L(\alpha_0)\}$$

is finite. $S \cap R$ is also non-empty, by virtue of containing α_0. In particular, there is a unique non-empty finite subset $\{\alpha_1, \ldots, \alpha_r\}$ of S such that

$$L(\alpha_i) = L(\alpha_j), \forall i, j$$

and

$$L(\alpha_i) < L(\alpha), \forall \alpha \in S - \{\alpha_1, \ldots, \alpha_r\}.$$

If $r = 1$, then α_1 is a vertex of S. Suppose $r > 1$. Then by perturbing L slightly to some $L + \epsilon$, one can arrange that, for some i,

$$(L + \epsilon)(\alpha_i) < (L + \epsilon)(\alpha), \forall \alpha \in S - \{\alpha_i\}.$$

So α_i is a vertex. Moreover, ϵ can be chosen so that $L - \epsilon$ is also minimized at a vertex. Since $\epsilon(\alpha_i) < \epsilon(\alpha_j)$ for all $j \neq i$, the vertex α_j where $L - \epsilon$ is minimized cannot be α_i. Now $\{\alpha_i, \alpha_j\}$ is the desired edge on which L is minimized. $\square$

We will now restrict to the case where Λ admits an action of $\mathcal{O}$. Let $\mathfrak{a}$ be a projective rank 1 $\mathcal{O}$-module and $\mathfrak{c}$ a fractional ideal of $\mathcal{O}$. Let $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. Then $\mathfrak{a}\mathfrak{b}$ is a lattice in $\mathfrak{a}\mathfrak{b} \otimes_{\mathbb{Z}} \mathbb{R}$.

If $\mathfrak{i}$ is any fractional ideal of $\mathcal{O}$, the real embeddings $\rho_1, \ldots, \rho_d$ of E induce an isomorphism

$$\rho_{\mathfrak{i}} : \mathfrak{i} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^d.$$

A trace form

$$\mathrm{Tr} : \mathfrak{i} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$$

is obtained by composing the trace form on $\mathbb{R}^d$ (sum of coordinates) with $\rho_{\mathfrak{i}}$. Likewise, a norm form

$$\mathbf{N} : \mathfrak{i} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$$

is obtained from the norm form on $\mathbb{R}^d$ (product of coordinates.)

A linear form $L : \mathfrak{a}\mathfrak{b} \to \mathbb{R}$ may be thought of as an element x_L of $\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying

$$L(\alpha) = \mathrm{Tr}(x_L \alpha).$$

A positive linear form corresponds to an element of $(\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++}$.

Fix a finite-index subgroup V of U^{++}, and let S be a subset of $\mathfrak{a}\mathfrak{b}^{++}$ which is stable under multiplication by V. The vertices and edges of such sets are rather manageable, thanks to the boundedness statement below and its corollaries.

Proposition 2.4. *Let D_V be the region*

$$\{x \in (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++} : \mathrm{Tr}(x) \leq \mathrm{Tr}(xu), \forall u \in V\}$$

and let A be a positive real number. Then the set

$$\{x \in D_V : \mathbf{N}(x) \leq A\}$$

is bounded.

Proof. Since D_V is invariant under multiplication by positive reals, it suffices to prove that

$$D_V^{(1)} = \{x \in D_V : \mathbf{N}(x) = 1\}$$

is bounded.

Composing $\rho_{\mathfrak{d}^{-1}}$ with the map

$$(x_1, \dots, x_d) \mapsto (\log |x_1|, \dots, \log |x_d|)$$

yields an isomorphism

$$\psi : (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++} \to \mathbb{R}^d$$

which induces a homeomorphism between the set of x in $(\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++}$ with $\mathbf{N}(x) = 1$ and the hyperplane

$$H = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_i x_i = 0\}.$$

If $\psi(x) = (y_1, \dots, y_d)$, we have

$$\begin{aligned} \mathrm{Tr}(x) &= \sum_i e^{y_i} \\ \psi(xu) &= (y_1 + \log \rho_1(u), \dots, y_d + \log \rho_d(u)), \forall u \in V. \end{aligned}$$

By Dirichlet's Unit Theorem, the points

$$\{\lambda_u = (\log \rho_1(u), \dots, \log \rho_d(u)) : u \in V\}$$

form a lattice in H, which we call Λ_V.

Choose a real number M large enough so that the region

$$\{(y_1, \dots, y_d) \in H : \max y_i \leq M\} \tag{2.1.4}$$

contains a fundamental domain for Λ_V. Now suppose $\psi(x) = (y_1, \dots, y_d)$ lies outside the bounded region

$$\{(y_1, \dots, y_d) \in H : \max y_i \leq M + \log d\}.$$

Then

$$\mathrm{Tr}(x) > e^{\max y_i} \geq de^M.$$

On the other hand, there exists $u \in V$ such that $h + \lambda_u$ lies inside (2.1.4), whence $\mathrm{Tr}(xu) \leq de^M$. We conclude that $x \notin D_V$, which proves the proposition. $\square$

Note that D_V is a fundamental domain for the action of V on $(\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++}$.

Corollary 2.5. *Let $\alpha_0 \in \mathfrak{ab}^{++}$ and $A \in \mathbb{R}^+$. There exist only finitely many $\alpha \in \mathfrak{ab}^{++}$ such that*

- $\mathbf{N}(\alpha_0^{-1}\alpha) \leq A$*;*
- *There exists a linear form L satisfying*

$$L(\alpha_0) \leq L(\alpha_0 u), \forall u \in V$$
$$L(\alpha) \leq L(\alpha u), \forall u \in V.$$

Proof. The second condition says that the intersection

$$\alpha_0^{-1} D_V \cap \alpha^{-1} D_V \in (\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes \mathbb{R})^{++}$$

is non-empty (by virtue of containing x_L), which is to say that

$$\alpha \in \alpha_0 D_V (D_V)^{-1}.$$

Adding the first condition, we have

$$\alpha_0^{-1}\alpha \in \{x \in D_V(D_V)^{-1} : \mathbf{N}(x) \leq A\}.$$

Let P be the set of all x in $D_V(D_V)^{-1}$ with norm at most A. Every element $x \in P$ can be written as $ry_1y_2^{-1}$, where $r \in (0, A]$ and $y_1, y_2 \in D_V^{(1)}$. By Proposition 2.4, $D_V^{(1)}$ is bounded, whence so is P. Since $\mathfrak{ab}^{++}$ is discrete, there are only finitely many α in $\mathfrak{ab}^{++}$ such that $\alpha_0^{-1}\alpha$ lies in P. The desired result follows. □

Corollary 2.6. *Let α_0 be an element of $(\mathfrak{ab} \otimes \mathbb{R})^{++}$. There exists $A \in \mathbb{R}$ such that $\mathbf{N}(\alpha_0^{-1}\alpha) \leq A$ for all vertices α of S.*

Proof. Without loss of generality, assume $\alpha_0 \in S$.

By Proposition 2.4, the set

$$\alpha_0^{-1} D_V^{(1)} \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes \mathbb{R}$$

is bounded, and in particular $L(\alpha_0)$ is bounded above by some constant C as L ranges over $\alpha_0^{-1} D_V^{(1)}$.

Now let α be a vertex of S, minimizing some linear form L. Since D_V is a fundamental domain for the action of V on $(\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++}$, we can replace L by Lu, for some $u \in V$, in such a way that $x_L \in \alpha_0^{-1} D_V$.

Now, multiplying L by a positive real number, we may suppose $x_L \in \alpha_0^{-1}(D_V)^{(1)}$. We replace α with αu^{-1}, so that α is still minimized at L.

Now $L(\alpha) \leq L(\alpha_0) \leq C$. But now

$$C \geq L(\alpha) = \mathrm{Tr}(x_L \alpha) \geq d\mathbf{N}(x_L\alpha)^{1/d} = d\mathbf{N}(\alpha_0^{-1}\alpha)^{1/d},$$

whence the desired result. □

Corollary 2.7. *Let α_0 be a vertex of S. Then there are only finitely many edges of S containing α_0.*

Proof. Let $\{\alpha_0, \alpha\}$ be an edge minimizing a linear form L. Then the second condition of Corollary 2.5 is met. By Corollary 2.6, the first condition is also met (for some A depending only on α_0.) Therefore, there are only finitely many possibilities for α. □

Corollary 2.8. *The actions of V on the vertices and on the edges of S have only finitely many orbits.*

Proof. Fix $\alpha_0 \in \mathfrak{ab}^{++}$, and choose a positive real number A such that $\mathbf{N}(\alpha_0^{-1}\alpha) \leq A$ for all vertices α of S. Now let α be a vertex of S. There exists $u \in V$ such that $\alpha u \in \alpha_0 D_V$. But it follows from Proposition 2.4 that the set of $x \in \alpha_0 D_V$ such that $N(\alpha_0^{-1}x) \leq A$ is bounded. So the set

$$T = \{x \in \mathfrak{ab}^{++} \cup \alpha_0 D_V : N(\alpha_0^{-1}x) \leq A\}$$

is finite, and we have shown that every vertex α is in the V-orbit of an element of T.

Moreover, every edge $\{\alpha_1, \alpha_2\}$ is in the orbit of some edge of the form $\{\alpha_1', \alpha_2'\}$, with $\alpha_1' \in T$. By Corollary 2.7, there are only finitely many such edges. The desired statement on the edges of S follows. □

Definition 2.9. Let $S_1, \ldots, S_r$ be subsets of $\mathfrak{ab}^{++}$ stable under V. A *compatible family* for $(S_1, \ldots, S_r)$ is an r-tuple $(\phi_1, \ldots, \phi_r)$, where ϕ_i is either a vertex or an edge of S_i, and there exists a linear form L which is minimized on every ϕ_i.

Proposition 2.10. *Let $S_1, \ldots, S_r$ be subsets of $\mathfrak{ab}^{++}$ invariant under V. Then the set of compatible families for $(S_1, \ldots, S_r)$ is acted on by V, and this action has only finitely many orbits.*

Proof. The existence of the action is apparent.

By Corollary 2.8, there are only finitely many choices for ϕ_1 up to the action of V. Fix such a choice. By Corollary 2.6, there exist $A_2, \ldots, A_r$ such that, if α_i is a vertex of S_i, we have $\mathbf{N}(\alpha_1^{-1}\alpha_i) \leq A_i$. Let α_0 be a vertex contained in ϕ_1 (e. g. $\alpha_0 = \phi_1$ if ϕ_1 is a vertex.) Similarly, let α be a vertex contained in ϕ_i. It now follows from Corollary 2.5 that there are only finitely many choices for α, whence only finitely many choices for ϕ_i. The proposition follows. □

2.2 The main theorem

We maintain the definitions of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ from the previous section. Let f be a $\mathfrak{c}$-cusp form of level N, defined over an integral domain R. Let $\operatorname{Spec} R_1, \ldots, \operatorname{Spec} R_m$ be an open affine cover of $\operatorname{Spec} R$ such that each ring R_k admits an isomorphism

$$j_k : \mathfrak{a} \otimes_{\mathbb{Z}} R_k \xrightarrow{\sim} \mathfrak{o} \otimes_{\mathbb{Z}} R_k.$$

Let

$$\epsilon : \mathfrak{o}/N\mathfrak{o} \xrightarrow{\sim} \mathfrak{a}/N\mathfrak{a}$$

be an isomorphism.

Then f has a q-expansion at the cusp $(\mathfrak{a}, \mathfrak{b}, j_k, \epsilon)/R_k$ of the form

$$f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega(j_k), \iota(\epsilon)) = \sum_{\alpha \in (\mathfrak{a}\mathfrak{b})^{++}} a_{k;\alpha} q^{\alpha}.$$

Define a set

$$S_{\mathfrak{a},\epsilon}(f) = \{\alpha \in \mathfrak{a}\mathfrak{b}^{++} : a_{k;\alpha} \neq 0\}.$$

This set is evidently independent of the choice of k and j_k. By Proposition 1.18, $S_{\mathfrak{a},\epsilon}(f)$ is invariant under the action of units of the form v^2, where $v \in U$ is congruent to 1 mod N and $\mathbf{N}^E_{\mathbb{Q}}(v) = 1$. Such units form a subgroup V of finite index in U^{++}. Thus, the definitions and results in the previous section may be applied to $S_{\mathfrak{a},\epsilon}(f)$.

Definition 2.11. Let f, R, ϵ be as above. We say f has *property (U)* at $(\mathfrak{a}, \epsilon)$ if $a_{k;\alpha} \in R_k^*$ for all k and all vertices α of $S_{\mathfrak{a},\epsilon}(f)$.

Suppose ϕ is a vertex or edge of a set $S \subset \mathfrak{a}\mathfrak{b}^{++}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are defined as in the previous section. Define

$$\delta(\phi) = \begin{cases} \alpha & \phi \text{ is a vertex } \alpha \\ \alpha - \alpha' & \phi \text{ is an edge } \{\alpha, \alpha'\} \end{cases}$$

The ambiguity in sign in the latter case will not concern us.

Definition 2.12. Suppose $(f_1, \ldots, f_r)$ is an r-tuple of $\mathfrak{c}$-cusp forms such that the following properties hold.

- f_m has property (U) at $(\mathfrak{a}, \epsilon)$, for all m in $\{1 \ldots r\}$.
- Suppose $(\phi_1, \ldots, \phi_r)$ is a compatible family for $(S_{\mathfrak{a},\epsilon}(f_1), \ldots, S_{\mathfrak{a},\epsilon}(f_r))$. Then the values $(\delta(\phi_1), \ldots, \delta(\phi_r))$ generate $\mathfrak{a}\mathfrak{b} \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space.

Then we say $\{f_1, \ldots, f_r\}$ is a *discriminantal set of cusp forms at* $(\mathfrak{a}, \epsilon)$. If $\{f_1, \ldots, f_r\}$ is a discriminantal set for every choice of $\mathfrak{a}$ and ϵ, we say simply that $\{f_1, \ldots, f_r\}$ is a *discriminantal set of cusp forms.*

Remark 2.13. To check whether a given $\{f_1, \ldots, f_r\}$ is a discriminantal set is a finite computation, since by Proposition 2.10 there are only finitely many V-orbits of compatible families for $(S_{\mathfrak{a},\epsilon}(f_1), \ldots, S_{\mathfrak{a},\epsilon}(f_r))$. (Here V is chosen so that multiplication by $u \in V$ stabilizes $S_{\mathfrak{a},\epsilon}(f_i)$ for all i.)

We are now ready for our main results.

Let $\{f_1, \dots, f_r\}$ be an arbitrary set of modular forms. For each compatible family $\boldsymbol{\phi} = (\phi_1, \dots, \phi_r)$ for $(S_{\mathfrak{a},\epsilon}(f_1), \dots, S_{\mathfrak{a},\epsilon}(f_r))$, we can define

$$M_{\boldsymbol{\phi}} = \bigoplus_i \mathbb{Z}\delta(\phi_i) \subset \mathfrak{ab}.$$

Let $M_{\boldsymbol{\phi}}^\vee \supset \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$ be the $\mathbb{Z}$-dual of $M_{\boldsymbol{\phi}}$.

Theorem 2.14. *Let $\{f_1, \dots, f_r\}$ be a set of $\mathfrak{c}$-Hilbert modular forms of level N over an integral domain R. Suppose*

- *A is a complete dvr with an R-algebra structure and fraction field K;*
- *$(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is a cusp over A;*
- *$q : \mathfrak{ab} \to \mathbb{G}_m(K)$ is a positive homomorphism.*

Let X/K be the HBAV

$$(\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{\mathfrak{a}}(\mathfrak{b}),$$

and let n be a rational integer such that

$$n | \mathrm{o}rd(f_i(X, \lambda_{can}, \omega(j), \iota(\epsilon)))$$

for all $i \in 1, \dots, r$. Then

$$\mathrm{o}rd(q) \in nM_{\boldsymbol{\phi}}^\vee$$

for some compatible family $\boldsymbol{\phi}$ for $(S_{\mathfrak{a},\epsilon}(f_1), \dots, S_{\mathfrak{a},\epsilon}(f_r))$.

Proof. Choose one of the R_k such that the image of $\operatorname{Spec} A$ in $\operatorname{Spec} R$ lies in $\operatorname{Spec} R_k$. Without loss of generality, suppose $R = R_k$.

We have

$$f_i(X, \lambda_{can}, \omega(j), \iota(\epsilon)) = \sum_{\alpha \in S_{\mathfrak{a},\epsilon}(f_i)} a_\alpha q(\alpha)$$

where the Fourier coefficients a_α are understood to be base changed from R to A.

Let $L : \mathfrak{ab} \to \mathbb{Z}$ be the positive linear form such that $x_L = \mathrm{ord}(q)$. Let ϕ_i be a vertex or edge of $S_{\mathfrak{a},\epsilon}(f_i)$ minimizing L; then $(\phi_1, \dots, \phi_r)$ is a compatible family for $(S_{\mathfrak{a},\epsilon}(f_1), \dots, S_{\mathfrak{a},\epsilon}(f_r))$.

Suppose ϕ_i is a vertex. Then $\mathrm{ord}(a_{\phi_i}) = 0$, by property (U), and

$$\mathrm{ord}(q(\phi_i)) = L(\phi_i) < L(\alpha) = \mathrm{ord}(q(\alpha))$$

for all $\alpha \in S_{\mathfrak{a},\epsilon}(f_i) - \{\phi_i\}$. So

$$\mathrm{ord}(f_i(X, \lambda_{can}, \omega(j), \iota(\epsilon))) = L(\phi_i) = L(\delta(\phi_i)).$$

Now suppose ϕ_i is an edge $\{\alpha, \alpha'\}$. Then $L(\alpha) = L(\alpha')$, and we have $L(\delta(\phi_i)) = 0$.

We have shown that $n|L(\delta(\phi_i))$ for all i. Therefore, $n|\mathrm{Tr}(\alpha \cdot \mathrm{ord}(q))$ for all $\alpha \in M_{\boldsymbol{\phi}}$. The proposition follows. ☐

Corollary 2.15. *Let $\{f_1, \ldots, f_r\}$ be a discriminantal set of $\mathfrak{c}$-Hilbert modular forms over level N over an integral domain R. Then there exists an integer $m_{\{f_i\}}$ such that, for every choice of A, n, q, X as above,*

$$\mathit{ord}(q) \in n'\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$$

where n' is the numerator of $n/m_{\{f_i\}}$ expressed in lowest terms.

Proof. Since $\{f_1, \ldots, f_r\}$ is discriminantal, every $M_{\boldsymbol{\phi}}$ is a finite-index subgroup of $\mathfrak{ab}$. So we can choose $m_{\{f_i\}}$ such that

$$m_{\{f_i\}}\mathfrak{ab} \subset M_{\boldsymbol{\phi}}$$

for each compatible family $\boldsymbol{\phi}$. The result then follows immediately from Theorem 2.14. ☐

The following corollary is an HBAV analogue of Corollary 1.2.

Corollary 2.16. *Let $\{f_1, \ldots, f_r\}$ be a discriminantal set of $\mathfrak{c}$-Hilbert modular forms of level N over an integral domain R. Suppose*

- *A is a complete dvr with an R-algebra structure and fraction field K;*
- *X/K is an HBAV with multiplicative reduction over A, endowed with a $\mathfrak{c}$-polarization λ, a Néron non-vanishing differential ω, and a Néron N-level structure ι.*

Let n be an integer such that

$$n|\mathit{ord}(f_i(X, \lambda, \omega, \iota))$$

for all $i \in 1, \ldots, r$, and let n' be the numerator of $n/m_{\{f_i\}}$, expressed in lowest terms. Then the group scheme $X[n']/K$ extends to a finite flat group scheme over A.

Proof. First of all, after an etale extension we may assume the reduction to be split multiplicative.

Now, by Theorem 1.19, we have

$$X \cong (\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{X;\mathfrak{a}}(\mathfrak{b}),$$

as $\mathfrak{c}$-polarized HBAV's, for some positive homomorphism q_X and some choice of $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. By Proposition 1.23, there is a cusp $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ over A such that $\iota = \iota(\epsilon)$ and $\omega = \omega(j)$.

We are now in the situation of Corollary 2.15. The desired result follows from Proposition 1.25. ☐

Remark 2.17. For any particular $\mathcal{O}$ and $f_1, \dots, f_r$, one can carry out a finer analysis. If the compatible family at which the linear form $\operatorname{Tr}(\alpha \cdot \operatorname{ord}(q_X))$ is minimized is known, then the ideal $\operatorname{ord}(q_X)\mathcal{O}$ can be calculated exactly in terms of the values of the $\operatorname{ord}(f_i)$. Since there are, up to units, only finitely many possibilities for the compatible family, there are only finitely many possibilities for $\operatorname{ord}(q_X)\mathcal{O}$ given the values $\operatorname{ord}(f_i)$. We will give examples of such computations in section 2.4.

2.3 Existence of discriminantal sets

Since $\mathfrak{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-vector space of dimension d, a discriminantal set must contain at least d modular forms. In this section we will show that a discriminantal set of this cardinality actually exists.

We begin by recalling some facts about HBAV's over $\mathbb{C}$. A *lattice* $\mathcal{L}$ in $E \otimes_{\mathbb{Q}} \mathbb{C}$ is a locally free rank 2 $\mathcal{O}$-submodule, and a $\mathfrak{c}$-polarization for $\mathcal{L}$ is an isomorphism

$$\Lambda : \wedge^2_{\mathcal{O}} \mathcal{L} \xrightarrow{\sim} \mathfrak{d}^{-1}\mathfrak{c}^{-1}$$

which is positive in the following sense: there is an $A \in (E \otimes_{\mathbb{Q}} \mathbb{R})^{++}$ such that

$$\Lambda(u, v) = \operatorname{Im}(\bar{u}v)/A.$$

There is a bijective correspondence between $\mathfrak{c}$-polarized lattices $(\mathcal{L}, \Lambda) \in E \otimes_{\mathbb{Q}} \mathbb{C}$ and triples $(X/\mathbb{C}, \lambda, \omega)$, where X is an HBAV, λ a $\mathfrak{c}$-polarization, and ω a non-vanishing differential [10, (1.4.6)]. An N-level structure ι on X corresponds to an injection

$$I : (1/N)\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \hookrightarrow (1/N)\mathcal{L}/\mathcal{L}.$$

Now let f be a modular form in $M^k_{\mathcal{O}}(\mathfrak{c}, 1; \mathbb{C})$, and let ν be a totally positive element of $\mathcal{O}$. Suppose $(\mathcal{L}, \Lambda, I)$ is a $\mathfrak{c}$-polarized lattice with N-level structure, $\nu | N$. Choose ℓ' in the image of I such that $\nu\ell' = 0$. Let $\mathcal{L}'$ be the lattice generated by $\mathcal{L}$ and ℓ'; note that $\mathcal{L}'$ is independent of the choice of ℓ'. Then extending Λ by bilinearity yields a $\mathfrak{c}$-polarization Λ' for $\mathcal{L}'$. Now define a function $B_\nu f$ on the space of $\mathfrak{c}$-polarized lattices with N-level structure by the rule

$$B_\nu f(\mathcal{L}, \Lambda, I) = \nu^{-k} f(\mathcal{L}', \Lambda').$$

Then $B_\nu f$ lies in $M^k_{\mathcal{O}}(\mathfrak{c}, N; \mathbb{C})$ by [10, (1.6.3)]. If the q-expansion of f at a cusp $(\mathfrak{a}, \mathfrak{b}, j, \epsilon_1)$ is

$$a_0 + \sum_{\alpha \in (\mathfrak{ab})^{++}} a_\alpha q^\alpha,$$

it follows from [10, (1.7.6)] that the q-expansion of $B_\nu f$ at $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$ is

$$a_0 + \sum_{\alpha \in (\mathfrak{ab})^{++}} a_\alpha q^{\nu\alpha}$$

for any choice of ϵ.

Suppose f is in $M^k_{\mathcal{O}}(\mathfrak{c}, 1; R)$, where R is a ring with an injection $R \hookrightarrow \mathbb{C}$. Then we can define $B_\nu f$ as a modular form over $\mathbb{C}$, which then descends to $M^k_{\mathcal{O}}(\mathfrak{c}, N; R)$ by the q-expansion principle [10, (1.2.16)].

Note that $B_\nu f$ has property (U) if and only if f does, and $S_{\mathfrak{a},\epsilon}(B_\nu f) = \nu S_{\mathfrak{a},\epsilon_1}(f)$ for all $\mathfrak{a}, \epsilon$.

Theorem 2.18. *Let $\mathfrak{c}$ be a fractional ideal of $\mathcal{O}$. Then there exists an integer m, a $\mathfrak{c}$-cusp form f of level 1, and a d-tuple $(\nu_1, \ldots, \nu_d)$ of elements of $\mathcal{O}^{++}$ such that $(B_{\nu_1} f, \ldots, B_{\nu_d} f)$ is a discriminantal set of modular forms over $\mathbb{Z}[1/m]$.*

Proof. First, there exists a $\mathfrak{c}$-cusp form f of level 1 defined over $\mathbb{Z}$. This follows, for instance, from [23, IV,4.4].

Let $\{f_1, \ldots, f_r\}, r < d$, be a set of modular forms such that

(P) for each choice of $(\mathfrak{a}, \epsilon)$, and for each compatible family $(\phi_1, \ldots, \phi_r)$ for $(S_{\mathfrak{a},\epsilon}(f_1), \ldots, S_{\mathfrak{a},\epsilon}(f_r))$, the values $(\delta(\phi_1) \ldots \delta(\phi_r))$ are linearly independent over $\mathbb{Q}$.

Note that a set of d modular forms with property (P) and property (U) is a discriminantal set.

Let V be a finite-index subgroup of U^{++} under which $S_{\mathfrak{a},\epsilon}(f_i)$ is invariant for all $\mathfrak{a}, \epsilon$, and i. Let $D_V \in (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{R})^{++} \xrightarrow{\sim} (E \otimes_{\mathbb{Z}} \mathbb{R})^{++}$ be the be the region defined in Proposition 2.4. We wil show there is some ν in $D_V \cap \mathcal{O}^{++}$ such that $(f_1, \ldots, f_r, B_\nu f_1)$ satifies (P).

Suppose ν is an element of $D_V \cap \mathcal{O}^{++}$ such that $\{f_1, \ldots, f_r, B_\nu f_1\}$ does not satisfy (P). Then there exists a compatible family $\{\phi_1, \ldots, \phi_r, \psi\}$ for some

$$(S_{\mathfrak{a},\epsilon}(f_1), \ldots, S_{\mathfrak{a},\epsilon}(f_r), S_{\mathfrak{a},\epsilon}(B_\nu f_1))$$

such that the values $(\delta(\phi_1), \ldots, \delta(\phi_r), \psi)$ are not linearly independent over $\mathbb{Q}$.

Let β be a vertex belonging to ψ, and for each i let α_i be a vertex belonging to ϕ_i. Since ψ and ϕ_1 simultaneously minimize some linear form, we know as in the proof of Corollary 2.5 that

$$\beta \in \alpha_1 (D_V)^{-1} D_V.$$

We have shown above that $S_{\mathfrak{a},\epsilon}(B_\nu f_1) = \nu S_{\mathfrak{a},\epsilon}(f_1)$. In particular, $\psi = \nu \phi'_1$ for some vertex or edge ϕ'_1 of $S_{\mathfrak{a},\epsilon}(f_1)$. Let α'_1 be the vertex belonging to ϕ'_1 such that $\beta = \nu \alpha'_1$. Then

$$\alpha'_1 = \beta \nu^{-1} \in \alpha_1 (D_V)^{-1} D_V (D_V)^{-1}.$$

Now, by assumption, $\delta(\psi) = \nu \delta(\phi'_1)$ lies in the $\mathbb{Q}$-vector space spanned by $\{\delta(\phi_1) \ldots \delta(\phi_r)\}$. Equivalently, ν lies in the $\mathbb{Q}$-vector space $W \subset \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by

$$\{\delta(\phi_1)\delta(\phi'_1)^{-1} \ldots \delta(\phi_r)\delta(\phi'_1)^{-1}\}.$$

We have shown in Proposition 2.10 that there are only finitely many choices, up to the action of V, for $(\phi_1, \dots, \phi_r)$. Suppose such a choice is fixed. Let $A \in \mathbb{R}$ be chosen such that $\mathbf{N}(\alpha_1^{-1}\alpha) \le A$ for all vertices α of $S_{\mathfrak{a},\epsilon}(f_1)$. By Proposition 2.4,

$$\{x \in (D_V)^{-1} D_V (D_V)^{-1} : \mathbf{N}(\alpha^{-1}x) < A\}$$

is bounded; therefore, there are only finitely many choices for α_1', whence only finitely many choices for ϕ_1'. There are also only finitely many choices for ϵ and the isomorphism class of $\mathfrak{a}$. Putting this all together, we have found that the set of ν for which $(f_1, \dots, f_r, B_\nu f_1)$ does not have property (P) is the intersection of $D_V \cap \mathcal{O}^{++}$ with a finite union of subspaces of $\mathcal{O} \otimes_Z \mathbb{Q}$ of dimension $i < d$. Thus we can choose ν such that $\{f_1, \dots, f_r, B_\nu f_1\}$ satisfies (P), as claimed.

Now proceed by induction. Evidently the set consisting solely of f satisfies (P), so we can produce a set of d cusp forms satisfying (P), defined over $\mathbb{Z}$. By passing to some $\mathbb{Z}[1/m]$, we can ensure that f_1, whence every $B_\nu f_1$, has property (U). We have now produced the desired discriminantal set of modular forms. □

2.4 Examples

In this section, we will consider the special cases $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ and $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$. In each case, we give an explicit discriminantal set of modular forms of level 1. The following proposition, a more explicit form of Corollary 2.6 in case $d = 2$, will be useful in the computations to follow.

Proposition 2.19. *Let $\mathcal{O}$ be the ring of integers of a real quadratic field. Let $\mathfrak{a}$ be a projective rank 1 $\mathcal{O}$-module, $\mathfrak{c}$ a fractional ideal of $\mathcal{O}$, and let $\mathfrak{b} = \mathfrak{a}\mathfrak{c}^{-1}$. Finally, let $S \subset \mathfrak{a}\mathfrak{b}^{++}$ be invariant under the action of V, a finite-index subgroup of U^{++} generated by v.*

Let α_0 be an element of S, and let α be a vertex of S. Then

$$\mathbf{N}(\alpha_0^{-1}\alpha) < (1/4)(2 + \mathrm{T}r(v)).$$

Proof. Let ρ_1, ρ_2 be the two embeddings of $\mathcal{O}$ into $\mathbb{R}$. Without loss of generality, suppose

$$\rho_2(v) > 1 > \rho_1(v).$$

Any element β of $(\mathfrak{a}\mathfrak{b} \otimes_{\mathbb{Z}} \mathbb{Q})^{++}$ can be written uniquely as

$$\beta = x_1 + x_2 v^{-1},$$

with $x_1, x_2 \in \mathbb{Q}$. We know that

$$\begin{aligned} \rho_2(v)/\rho_1(v) &> 1 \\ \rho_2(1)/\rho_1(1) &= 1 \\ \rho_2(v^{-1})/\rho_1(v^{-1}) &< 1. \end{aligned}$$

So $\rho_2(\beta)/\rho_1(\beta) < 1$ if and only if x_2 is positive. Likewise, $\rho_2(v\beta)/\rho_1(v\beta) > 1$ if and only if x_1 is positive.

Now let $\beta_n = v^n \alpha_0^{-1}\alpha$. For large n, we have $\rho_2(\beta_n)/\rho_1(\beta_n) > 1$, while for small n, the opposite is the case. So we may choose n such that

$$\rho_2(\beta_n)/\rho_1(\beta_n) < 1 < \rho_2(\beta_{n+1})/\rho_1(\beta_{n+1}). \tag{2.4.5}$$

Now write

$$\beta_n = x_1 + x_2 v^{-1};$$

by (2.4.5), x_1 and x_2 are positive. Now if $x_1 + x_2 > 1$, then $\alpha_0\beta_n = v^n\alpha$ is in the interior of the positive half-plane bounded by the line through α_0 and $v^{-1}\alpha_0$, and thus cannot be a vertex.

We conclude that

$$\beta_n = x_1 + x_2 v^{-1}, x_1, x_2 \in \mathbb{Q}_{>0}, x_1 + x_2 \leq 1;$$

the maximal norm on this region is $(1/4)(2 + \mathrm{Tr}(v))$, attained when $x_1 = x_2 = 1/2$.

□

Now let $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and let $\mathfrak{c} = \mathfrak{d}$. Recall from Example 1.11 the cusp forms χ_6 and χ_{12}, defined over $\mathbb{Z}$, of level 1.

Proposition 2.20. $\{\chi_6, \chi_{12}\}$ *is a discriminantal set of Hilbert modular forms over* $\mathbb{Z}$*, with* $m_{\{\chi_6,\chi_{12}\}} = 2$.

Proof. Since the class group of $\mathcal{O}$ is trivial, we may set $\mathfrak{a} = \mathfrak{o}$ and $\mathfrak{b} = \mathfrak{d}^{-1}$. Since the level of both forms is 1, we may set $\epsilon = \epsilon_1$. We therefore suppress subscripts and write $S(\chi_i)$ for $S_{\mathfrak{a},\epsilon}(\chi_i)$. Finally, take j to be the identity map on $\mathfrak{o}$. The q-expansions of χ_6 and χ_{12} can be

The computation of the q-expansions of χ_6 and χ_{12} at $(\mathfrak{o}, \mathfrak{d}^{-1}, j, \epsilon_1)$ is s-traightforward. First of all, it suffices to compute the Fourier expansions of χ_6 and χ_{12} as holomorphic Hilbert modular forms [10, (1.7.6)]. Then one can use either the well-known expressions of χ_6 and χ_{12} in terms of Eisenstein series or the method of Resnikoff [19] to compute these Fourier coefficients. One finds that

$$\begin{aligned} \chi_6 &= q^{\frac{1}{2}-\frac{\sqrt{5}}{10}} + q^{\frac{1}{2}+\frac{\sqrt{5}}{10}} + q^{1-\frac{2\sqrt{5}}{5}} + 20q^{1-\frac{\sqrt{5}}{5}} \\ &\quad -90q + 20q^{1+\frac{\sqrt{5}}{5}} + q^{1+\frac{2\sqrt{5}}{5}} + \dots \end{aligned}$$

and

$$\chi_{12} = q + \dots,$$

where in each case the terms $a_\alpha q^{\alpha)}$ with $\mathrm{Tr}(\alpha) > 2$ are omitted.

Since both forms have even weight, $S(f_i)$ is invariant under all of $U^2 = U^{++}$ by Proposition 1.18. This group is generated by $u = \frac{3+\sqrt{5}}{2}$.

It follows from Proposition 2.19 that every vertex of $S(\chi_6)$ is an element of $(1/2 - \sqrt{5}/10)U^{++}$. Similarly, every vertex of $S(\chi_{12})$ lies in U^{++}. From there, it is a simple matter to verify that the compatible families (ϕ_1, ϕ_2) for $(S(\chi_6), S(\chi_{12}))$ split into the following four orbits under U^{++}:

$$
\begin{gathered}
(1/2 - \sqrt{5}/10, 1) \\
(1/2 - \sqrt{5}/10, 3/2 - \sqrt{5}/2) \\
(1/2 - \sqrt{5}/10, \{1, 3/2 - \sqrt{5}/2\}) \\
(\{1/2 - \sqrt{5}/10, 1/2 + \sqrt{5}/10\}, 1)
\end{gathered}
$$

Each of these families constitutes a $\mathbb{Q}$-basis for $\mathbb{Q}[\sqrt{5}]$. By inspection of the q-expansions above, χ_6 and χ_{12} have property (U). So $\{\chi_6, \chi_{12}\}$ is a discriminantal set. Since the $\mathbb{Z}$-span of $\delta(\phi_1)$ and $\delta(\phi_2)$ contains $2\mathfrak{d}^{-1}$ for each family above, we have $m_{\chi_6,\chi_{12}} = 2$. □

Corollary 2.21. *Let X/K be a $\mathfrak{c}$-polarized HBAV with real multiplication by $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ and multiplicative reduction, λ its polarization, and ω a Néron non-vanishing differential. Let n be an integer such that*

$$
\begin{aligned}
n &\mid \mathrm{ord}(\chi_6(X, \lambda, \omega, \iota_1)) \\
n &\mid \mathrm{ord}(\chi_{12}(X, \lambda, \omega, \iota_1)).
\end{aligned}
$$

Then the group scheme $X[n']/K$ extends to a finite flat group scheme over A, where n' is the numerator of $n/2$ expressed in lowest terms.

Proof. Immediate from Proposition 2.20 and Corollary 2.16. □

Proposition 2.22. *Let X, λ, ω be as in Corollary 2.21, and set*

- $n_6 = \mathrm{ord}(\chi_6(X, \lambda, \omega, \iota_1))$;
- $n_{12} = \mathrm{ord}(\chi_{12}(X, \lambda, \omega, \iota_1))$.

Suppose furthermore that X has split multiplicative reduction, so that X is uniformized as

$$X \cong (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)/q_{X;\mathfrak{o}}(\mathfrak{d}^{-1}).$$

Then $\mathrm{ord}(q_X)$ is one of the values

$$
\begin{gathered}
-\sqrt{5}n_6 + [(1+\sqrt{5})/2]n_{12}, \\
\sqrt{5}n_6 + [(1-\sqrt{5})/2]n_{12}, \\
\left[(5+\sqrt{5})/4\right] n_6, \\
(1/2)n_{12}
\end{gathered}
$$

up to multiplication by a totally positive unit.

Proof. We divide the proposition into three cases. Let $L : \mathfrak{d}^{-1} \to \mathbb{Z}$ be the linear form such that $L(\alpha) = \mathrm{Tr}(\mathrm{ord}(q_X)\alpha)$.

Case 1: $\mathrm{ord}(q_X)$ is a rational integer, up to multiplication by a totally positive unit.

Without loss of generality, we may take $\mathrm{ord}(q_X)$ to be a rational integer m. It follows that L is minimized at the vertex $\alpha = 1$ of $S(\chi_{12})$. So

$$n_{12} = \mathrm{ord}(\chi_{12}(X, \lambda, \omega, \iota_1)) = \mathrm{Tr}(\mathrm{ord}(q_X)) = 2m.$$

Case 2: $\mathrm{ord}(q_X) = (\frac{5}{2} + \frac{\sqrt{5}}{2})mu$, where m is a rational integer and u is a totally positive unit.

Again, we may assume $\mathrm{ord}(q_X) = (\frac{5}{2} + \frac{\sqrt{5}}{2})m$. Then L is minimized at the vertex $\frac{1}{2} - \frac{\sqrt{5}}{10}$ of $S(\chi_6)$. So

$$n_6 = \mathrm{ord}(\chi_6(X, \lambda, \omega, \iota_1)) = \mathrm{Tr}\left(\left(\frac{1}{2} - \frac{\sqrt{5}}{10})(\frac{5}{2} + \frac{\sqrt{5}}{2}\right) m\right) = 2m.$$

Case 3: Neither Case 1 nor Case 2 obtains.

One then checks that L is not minimized on any edge of $S(\chi_6)$ or $S(\chi_{12})$. Thus, L is minimized on some vertex α_6 of $S(\chi_6)$ and on some vertex α_{12} of $S(\chi_{12})$. By the list of compatible families for $(S(\chi_6), S(\chi_{12}))$ given in the proof of Proposition 2.20, this implies that, after modification by a totally positive unit, (α_6, α_{12}) is either $(1/2 - \sqrt{5}/10, 1)$ or $(1/2 - \sqrt{5}/10, 3/2 - \sqrt{5}/2)$. Suppose $(\alpha_6, \alpha_{12}) = (1/2 - \sqrt{5}/10, 1)$. We then have

$$\begin{aligned} n_6 = \mathrm{ord}(\chi_6(X, \lambda, \omega, \iota_1)) &= \mathrm{Tr}\left(\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right) \cdot \mathrm{ord}(q_X)\right) \\ n_{12} = \mathrm{ord}(\chi_{12}(X, \lambda, \omega, \iota_1)) &= \mathrm{Tr}(\mathrm{ord}(q_X)). \end{aligned}$$

It follows that

$$\mathrm{ord}(q_X) = -\sqrt{5}n_6 + [(1 + \sqrt{5})/2]n_{12}.$$

Observe that $(1/2 - \sqrt{5}/10, 3/2 - \sqrt{5}/2)$ differs by a totally positive unit from $(1/2 + \sqrt{5}/10, 1)$. Assuming that $(\alpha_6, \alpha_{12}) = (1/2 + \sqrt{5}/10, 1)$ then yields

$$\mathrm{ord}(q_X) = \sqrt{5}n_6 + [(1 - \sqrt{5})/2]n_{12}.$$

The desired result is now proven. □

Example 2.23. Let $X/\mathbb{Q}$ be the Jacobian of the genus 2 curve

$$C : y^2 = f(x) = (x^3 - 4x^2 - 3x - 1)(x^3 + x + 3).$$

Then X is an HBAV with real multiplication by $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, since it is the specialization to $b=-4, c=-2, d=-1$ of Brumer's family (1.2.2). The only primes dividing the discriminant of f are $2, 13$ and 19, so X has good reduction away from these primes.

The reduction of the given model of C at 13 is the stable curve

$$y^2 = (x-2)^2(x-4)^2(x-9)^2,$$

so X has semistable reduction at 13. We can compute the values of the Hilbert modular forms χ_6 and χ_{12} at X as follows. First, we use Igusa's formulas in [9] to compute the values of Siegel modular functions in terms of the projective invariants of the sextic f. Then, we use Resnikoff's calculations in [19] to compute the values of Hilbert modular functions of weight 0. (Note that in the weight 0 case, we need not specify a choice of ω.) We find

$$\begin{aligned} \chi_6\phi_2^{-3}(X,\lambda) &= 5 \cdot 13^2 \cdot 19 \cdot 79^{-3} \\ \chi_{12}\phi_2^{-6}(X,\lambda) &= 2 \cdot 13^5 \cdot 19^2 \cdot 79^{-6} \end{aligned}$$

where ϕ_2 is the modular form over $\mathbb{Z}$ described in Example 1.11, and λ is the unique polarization (up to isomorphism) on X. One can compute that every q-expansion of ϕ_2 has nonzero constant coefficient [19]; thus,

$$\operatorname{ord}_{13}(\phi_2(X,\lambda,\omega,\iota_1)) = 0$$

for every Néron non-vanishing differential on X. So

$$\begin{aligned} \operatorname{ord}_{13}(\chi_6(X,\lambda,\omega,\iota_1)) &= 2 \\ \operatorname{ord}_{13}(\chi_{12}(X,\lambda,\omega,\iota_1)) &= 5. \end{aligned}$$

Let q_X be the Tate parameter of $X/\mathbb{Z}_{13}$. It follows from Proposition 2.22 that

$$\operatorname{ord}_{13}(q_X) = (5+\sqrt{5})/2. \tag{2.4.6}$$

The model

$$C' : y^2 + y(x^3+x+1) = -x^5 - x^4 - x^3 - 4x^2 - 3x - 1$$

for C is smooth over $\mathbb{Z}_2$, so X has good reduction at 2. Also, one observes that X is semistable at 19.

Suppose the Galois representation on $X[\sqrt{5}]$

$$\rho : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{F}_5)$$

is irreducible. Then, by [21, Th. 4.2], X is modular. There is a unique modular form f of weight 2 and level $13 \cdot 19 = 247$ whose Fourier coefficients are defined over $\mathbb{Q}[\sqrt{5}]$[1]; therefore, X is isogenous to the corresponding factor X_f of $J_0(247)$.

[1] This can be seen from the tables of Fourier coefficients recently produced by William Stein, available at `http://www.math.berkeley.edu/~was`.

Moreover, it follows from (2.4.6) that ρ is unramified at 13; thus, ρ has conductor 19. So ρ must be the Galois representation associated to g, where g is the unique cuspform of weight 2 and level 19. But one observes by comparing Fourier coefficients that f and g are not congruent mod 5. Therefore, ρ is reducible, which is to say that $X[\sqrt{5}]$ possesses a rational cyclic subgroup of order 5. By counting points over finite fields we can verify that the semisimplification of ρ is the direct product of the trivial character with the cyclotomic character; in other words, either X or an HBAV $\sqrt{5}$-isogenous to X contains a rational $\sqrt{5}$-torsion point.

Now suppose $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$. Once again, the narrow class group is trivial, so we may set $\mathfrak{c} = \mathfrak{d}$. There are $\mathfrak{d}$-cusp forms over $\mathbb{C}$ of level 1 and weights 4 and 6 ([8]), which were shown by Nagaoka [15] to descend to forms χ_4 and χ_6 over $\mathbb{Z}$.

Proposition 2.24. *$\{\chi_4, \chi_6\}$ is a discriminantal set of modular forms over $\mathbb{Z}$, with $m_{\{\chi_4,\chi_6\}} = 2$.*

Proof. As in the proof of Proposition 2.20, we may set $\mathfrak{a} = \mathfrak{o}$, $\mathfrak{b} = \mathfrak{d}^{-1}$, $j = id$, and $\epsilon = \epsilon_1$, and refer simply to $S(\chi_4)$ and $S(\chi_6)$. Müller [13] gave expressions for χ_4 and χ_6 in terms of theta functions, from which we can derive the q-expansions at $(\mathfrak{a}, \mathfrak{b}, j, \epsilon)$

$$\begin{aligned}\chi_4 &= q^{1/2-\sqrt{2}/4} - 2q^{1/2} + q^{1/2-\sqrt{2}/4} \\ &\quad - 4q^{1-\sqrt{2}/2} - 8q^{1-\sqrt{2}/4} + 24q - 8q^{1+\sqrt{2}/4} - 4q^{1+\sqrt{2}/2} + \ldots \\ \chi_6 &= q^{1/2} + 2q^{1-\sqrt{2}/2} - 176q^{1-\sqrt{2}/4} - 684q - 176q^{1+\sqrt{2}/4} + 2q^{1+\sqrt{2}/2} + \ldots\end{aligned}$$

The omitted terms are of the form $a_\alpha q^\alpha$ with $\mathrm{Tr}(\alpha) > 2$.

Both $S(\chi_4)$ and $S(\chi_6)$ are invariant under the action of U^{++}, by Proposition 1.18. It follows from Proposition 2.19 that the vertices and edge-elements of $S(\chi_4)$ are the elements of $(1/2-\sqrt{2}/4)U^{++}$, and the vertices and edge-elements of $S(\chi_6)$ the elements of $(1/2)U^{++}$. It is then immediate that χ_4 and χ_6 have property (U). The compatible families (ϕ_1, ϕ_2) for $(S(\chi_4), S(\chi_6))$ split into the following four orbits under the action of U^{++};

$$\begin{gathered}(1/2 - \sqrt{2}/4, 1/2) \\ (1/2 - \sqrt{2}/4, 3/2 - \sqrt{2}) \\ (1/2 - \sqrt{2}/4, \{3/2 - \sqrt{2}, 1/2\}) \\ (\{1/2 - \sqrt{2}/4, 1/2 + \sqrt{2}/4\}, 1/2)\end{gathered}$$

In each case, $(\delta(\phi_1), \delta(\phi_2))$ is a $\mathbb{Q}$-basis for $\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{Q}$, and the $\mathbb{Z}$-span of $\delta(\phi_1)$ and $\delta(\phi_2)$ contains $2\mathfrak{d}^{-1}$. □

Corollary 2.25. *Let X/K be a $\mathfrak{c}$-polarized HBAV with real multiplication by $\mathcal{O}$ and multiplicative reduction, λ its polarization, and ω a Néron non-vanishing differential. Let n be an integer such that*

$$\begin{aligned} n &\mid ord(\chi_4(X, \lambda, \omega, \iota_1)) \\ n &\mid ord(\chi_6(X, \lambda, \omega, \iota_1)).\end{aligned}$$

Then the group scheme $X[n']/K$ extends to a finite flat group scheme over A, where n' is the numerator of $n/2$ expressed in lowest terms.

Proof. Immediate from Proposition 2.24 and Corollary 2.16. □

From our description of the compatible families for $\{\chi_4, \chi_6\}$, we produce an analogue of Proposition 2.22.

Proposition 2.26. *Let X, K, λ, ω' be as above, and let*

- $n_4 | ord(\chi_4(X, \lambda, \omega', \iota_1))$
- $n_6 | ord(\chi_6(X, \lambda, \omega', \iota_1))$.

Then $ord(q_X)$ is one of the values

$$\begin{gathered} -\sqrt{2}n_4 + (1+\sqrt{2})n_6, \\ (4+3\sqrt{2})n_4 + (-1-\sqrt{2})n_6, \\ (2+\sqrt{2})n_4, \\ n_6 \end{gathered}$$

up to multiplication by a totally positive unit.

Proof. The proof is a computation exactly analogous to that in Proposition 2.22. □

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