

Endomorphism algebras of Jacobians

Jordan S. Ellenberg

*Department of Mathematics, Princeton University
Fine Hall, Washington Road
Princeton, NJ 08544*

E-mail: ellenber@math.princeton.edu

Very little is known about the existence of curves, and families of curves, whose Jacobians are acted on by large rings of endomorphisms. In this paper, we show the existence of curves X with an injection

$$K \hookrightarrow \mathrm{Hom}(\mathrm{Jac}(X), \mathrm{Jac}(X)) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where K is a subfield of even index at most 10 in a primitive cyclotomic field $\mathbb{Q}(\zeta_p)$, or a subfield of index 2 in $\mathbb{Q}(\zeta_{pq})$ or $\mathbb{Q}(\zeta_{p^\alpha})$. This result generalizes previous work of Brumer, Mestre, and Tautz-Top-Verberkmoes. Our curves are constructed as branched covers of the projective line, and the endomorphisms arise as quotients of double coset algebras of the Galois groups of these coverings. In certain cases, we show the existence of continuous families of Jacobians admitting the desired endomorphism algebras. At the end, we raise some questions about upper bounds for endomorphism algebras of the type described here, and ask about a “non-Galois version of Hurwitz’s theorem.”

2000 MSC classification: 14H40 (14H30)

Key Words: Jacobian, endomorphism, Hecke, double coset, Hurwitz.

INTRODUCTION

Let k be an algebraically closed field, A/k a principally polarized abelian variety, and $\mathrm{End}_0(A)$ the \mathbb{Q} -algebra of endomorphisms of A . It has long been known which algebras E arise as $\mathrm{End}_0(A)$ for some principally polarized abelian variety A [23, §21]. If we restrict our attention to Jacobians of smooth curves of a given genus, however, the question is less well understood. Van der Geer and Oort remark [28, §5]:

“...one expects excess intersection of the Torelli locus and the loci corresponding to abelian varieties with *very large* endomorphism rings; that is, one expects that they intersect much more than their dimensions suggest.”

This expectation has been borne out by many examples of families of curves whose Jacobians are acted on by large algebras. Theorems of Tautz-Top-Verberkmoes [27], Mestre [19], and Brumer [2] produce, for each odd prime p , families of curves X of genus $(p-1)/2$ such that $\text{End}_0(\text{Jac}(X))$ contains the real cyclotomic field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Brumer also produces a family of genus 2 curves with $\mathbb{Q}(\sqrt{2}) \subset \text{End}_0(\text{Jac}(X))$. Hashimoto-Murabayashi [15] and Bending [1] have produced families of genus 2 curves whose Jacobians are acted on by some quaternion algebras of small discriminant. De Jong and Noot give examples of infinitely many curves of genus 4 and 6 whose Jacobians have complex multiplication [4]. Ekedahl-Serre [9] produce many examples of curves of genus g with $\mathbb{Q}^{\oplus g} \subset \text{End}_0(\text{Jac}(X))$. Finally, Shimada [26] uses methods similar to ours to produce examples of curves of large genus whose Jacobians are acted on by quadratic fields.

In the present article we present a general procedure for constructing curves whose Jacobians have large endomorphism algebras, and we show that the endomorphisms produced in [27],[19], and [2], and most of those in [9], arise via our construction. Furthermore, we show the existence of some new families of curves whose Jacobians are acted on by totally real fields.

Let F be a totally real number field. We say that a curve X over a field k has *real multiplication* by F if $g(X) = [F : \mathbb{Q}]$ and if X admits an injection $F \hookrightarrow \text{End}_0(\text{Jac}(X))$.¹ We denote the index n subfield of $\mathbb{Q}(\zeta_p)$ by $\mathbb{Q}(\zeta_p^{(n)})$. Our main theorem is the following.

MAIN THEOREM. Let k be an algebraically closed field. Then

1. If $p > 5$ is a prime, and $\text{char } k$ does not divide $2p$, there exists a 3-dimensional family of curves of genus $(p-1)/2$ over k with real multiplication by $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$.

2. If p is a prime congruent to 1 mod 4, and $\text{char } k$ does not divide $2p$, then there exists a 1-dimensional family of curves of genus $(p-1)/4$ over k with real multiplication by $\mathbb{Q}(\zeta_p^{(4)})$.

3. If p is a prime congruent to 1 mod 6, and $\text{char } k$ does not divide $6p$, then there exists a 1-dimensional family of curves of genus $(p-1)/6$ over k with real multiplication by $\mathbb{Q}(\zeta_p^{(6)})$.

4. If p is a prime congruent to 1 mod 8, and $\text{char } k$ does not divide $2p$, then there exists a curve of genus $(p-1)/8$ over k with real multiplication by $\mathbb{Q}(\zeta_p^{(8)})$.

5. If p is a prime congruent to 1 mod 10, and $\text{char } k$ does not divide $10p$, then there exists a curve of genus $(p-1)/10$ over k with real multiplication by $\mathbb{Q}(\zeta_p^{(10)})$.

¹This notation is slightly non-standard in case $\text{char } k$ divides the discriminant of F/\mathbb{Q} ; see [8]. However, this case will not arise in the present work.

6. If p, q are distinct odd primes, and $\text{char } k$ does not divide $2pq$, there exists a 1-dimensional family of curves of genus $(p-1)(q-1)/2$ over k with real multiplication by $\mathbb{Q}(\zeta_{pq} + \zeta_{pq}^{-1})$.

7. If p is an odd prime, and $\text{char } k$ does not divide $2p$, and $\alpha > 1$ is an integer, there exists a 2-dimensional family of curves of genus $p^{\alpha-1}(p-1)/2$ over k with real multiplication by $\mathbb{Q}(\zeta_{p^\alpha} + \zeta_{p^\alpha}^{-1})$.

Let X_0/k be a smooth curve, and Y/X_0 a Galois cover with Galois group G . Let H be a subgroup of G and let X be the quotient curve Y/H . Then the Jacobian of X is acted on by the double coset algebra $\mathbb{Q}[H \backslash G / H]$. We say that the image of $\mathbb{Q}[H \backslash G / H]$ in $\text{End}_0(X)$ is of *Hecke type*. The study of endomorphism algebras of Hecke type is the main purpose of the present paper. In section 1, we explain how to compute the endomorphism algebra of Hecke type associated to a branched cover $X \rightarrow X_0$. We construct the families of curves referred to above in sections 2 and 3, and show that these families have the stated dimensions in section 4.

The negative side of the question—that is, the question of which algebras *cannot* occur in $\text{End}_0(\text{Jac}(X))$ —is more mysterious. One can also ask weaker questions about bounds on algebras which can occur as endomorphism algebras of Hecke type attached to smooth curves. We discuss the current state of knowledge about such questions in section 5.

NOTATION

If V is a vector space acted on by a group H , denote by V^H the subspace of V fixed by H .

A “curve” will always be a nonsingular projective algebraic curve over a field unless specifically declared otherwise.

If A is an abelian variety, $\text{End}_0(A)$ is its \mathbb{Q} -algebra of endomorphisms $\text{Hom}(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If p is a prime and n a divisor of $p-1$, we denote by $\mathbb{Q}(\zeta_p^{(n)})$ the subfield of index n in the cyclotomic field $\mathbb{Q}(\zeta_p)$. For any integer m , we write $\mathbb{Q}(\zeta_m^+)$ for the real cyclotomic field $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$.

If $X \rightarrow C$ is a dominant morphism of algebraic curves, the *Galois group of X/C* is understood to mean the group of the Galois covering Y/C , where Y is the Galois closure of X/C .

1. ENDOMORPHISM ALGEBRAS OF HECKE TYPE

Let Y be a nonsingular projective algebraic curve over an algebraically closed field k , and let G be a finite group which acts on Y . Write C for the

quotient of Y by G (Precisely, C is the nonsingular curve associated to the G -invariant subfield of $k(Y)$.)

The map

$$\phi : G \rightarrow \text{Aut}(Y)$$

induces a map

$$e : \mathbb{Q}[G] \rightarrow \text{End}_0(\text{Jac}(Y)).$$

Now let H be a subgroup of G , and let X be the quotient curve Y/H . Define $\pi_H \in \mathbb{Q}[G]$ to be

$$(1/|H|) \sum_{h \in H} h.$$

We define the *Hecke algebra* $\mathbb{Q}[H \backslash G / H]$ to be the subalgebra of $\mathbb{Q}[G]$ generated by $\pi_H g \pi_H$ for all $g \in G$.

Remark 1. 1. We recall some basic facts about representations of finite groups which we will need along the way. For proofs, see [18, 5.3]. The group algebra $\mathbb{Q}[G]$ admits a direct product decomposition

$$\mathbb{Q}[G] \cong \bigoplus_i A_i.$$

Each simple factor A_i is isomorphic to $M_{n_i}(\Delta_i)$, where Δ_i is a central simple algebra whose center K_i is an abelian extension of \mathbb{Q} . Let e_{jj} be the matrix in $M_{n_i}(\Delta_i)$ which has a 1 in the (j, j) position and zeroes elsewhere. Then $V_i := M_{n_i}(\Delta_i)e_{11}$ is an irreducible representation of $\mathbb{Q}[G]$, and in fact every irreducible representation of $\mathbb{Q}[G]$ arises as V_i for exactly one i . We can think of V_i as a vector space over Δ_i^{op} , in which case

$$A_i = \text{End}_{\Delta_i^{op}}(V_i).$$

In particular, π_H commutes with the action of Δ_i^{op} on V_i ; it follows that $V_i^H = \pi_H V_i$ is a sub- Δ_i^{op} -vector space of V_i , and that

$$\pi_H A_i \pi_H = \pi_H \text{End}_{\Delta_i^{op}}(V_i) \pi_H = \text{End}_{\Delta_i^{op}}(V_i^H) \cong M_{d_i}(\Delta_i)$$

for some $d_i \leq n_i$. So we have a decomposition

$$\mathbb{Q}[H \backslash G / H] \cong \bigoplus_i M_{d_i}(\Delta_i). \quad (1.1)$$

The action e of $\mathbb{Q}[G]$ restricts to a natural action

$$\mathbb{Q}[H \backslash G / H] \rightarrow \text{End}_0(e(\pi_H) \text{Jac}(Y)). \quad (1.2)$$

The natural map

$$\text{Jac}(X) \rightarrow \text{Jac}(Y)$$

restricts to an isogeny

$$\text{Jac}(X) \rightarrow e(\pi_H) \text{Jac}(Y).$$

Therefore, the action (1.2) yields an action

$$\mathbb{Q}[H \backslash G / H] \rightarrow \text{End}_0(\text{Jac}(X)).$$

We refer to the image of $\mathbb{Q}[H \backslash G / H]$ in $\text{End}_0(\text{Jac}(X))$ as an *endomorphism algebra of Hecke type*, and denote it by $\mathcal{H}_{X/C}$. In general, we say an element α of $\text{End}_0(\text{Jac}(X))$ is of Hecke type if there exists some branched cover $X \rightarrow C$ such that $\alpha \in \mathcal{H}_{X/C}$.

We note in passing that there exist Jacobians of smooth curves with endomorphisms which are not of Hecke type. The Jacobians of smooth genus 3 curves form an open subscheme of the moduli space of principally polarized abelian three-folds; in particular, there exist Jacobians of smooth genus 3 curves which have real multiplication by non-abelian cubic fields. Among these, choose a curve X whose Jacobian has no endomorphisms other than those coming from the cubic field.

Let $X \rightarrow C$ be some surjective maps of curves, with Galois group G . Now $\mathcal{H}_{X/C}$ is a quotient of $\mathbb{Q}[H \backslash G / H]$, and as such is a direct sum of central simple algebras over abelian extensions of \mathbb{Q} . In particular, $\mathcal{H}_{X/C}$ cannot be a non-abelian cubic extension of \mathbb{Q} . So $\text{Jac}(X)$ has endomorphisms which are not of Hecke type.

Let ℓ be a prime not equal to the characteristic of k . The natural map

$$\text{End}_0(\text{Jac}(X)) \rightarrow \text{End}_0(H_1(X, \mathbb{Q}_\ell))$$

is injective. Thus, to compute $\mathcal{H}_{X/C}$ it suffices to study the representation of $\mathbb{Q}[H \backslash G / H]$ on $H_1(X, \mathbb{Q}_\ell) \cong \pi_H H^1(Y, \mathbb{Q}_\ell)$. This representation, in turn, is determined by the action of G on $H^1(Y, \mathbb{Q}_\ell)$. Refer to this representation of G as ρ_Y , and denote its character by χ_Y . One can compute χ_Y from the branching data of the map $Y \rightarrow C$, by means of Proposition 1.1 below.

Notation: for each $g \in G$, denote by χ_g the character of G induced from the trivial character on the cyclic group $\langle g \rangle$. Write χ_{triv} for the trivial character of G .

Remark 1. 2. Let U be the open curve $C - \{p_1, \dots, p_r\}$. If $t \in k(C)$ is a uniformizer at p_i , then we have a morphism

$$\text{Spec } k((t)) \rightarrow U$$

which induces a map of étale fundamental groups

$$\text{Gal}(k((t))^{sep}/k((t))) \rightarrow \pi_1(U),$$

defined only up to conjugacy, since we have neglected to specify basepoints. The étale G -cover $Y \times_C U \rightarrow U$ allows us to extend this diagram to

$$\text{Gal}(k((t))^{sep}/k((t))) \rightarrow \pi_1(U) \rightarrow G.$$

When we say “ $Y \rightarrow C$ has monodromy g_i at p_i ,” we mean that the composite homomorphism above factors through the tame quotient of the Galois group $\text{Gal}(k((t))^{sep}/k((t)))$, and that the image of a generator of tame inertia lies in the conjugacy class of g_i . In case $k = \mathbb{C}$, this definition coincides with the natural topological one.

PROPOSITION 1.1. *Suppose the map $Y \rightarrow C$ is branched at r points $p_1, \dots, p_r \in C(k)$, with monodromy g_1, \dots, g_r , and suppose that each g_i has order prime to $\text{char } k$. Then*

$$\chi_Y = 2\chi_{triv} + 2(g(C) - 1)\chi_1 + \sum_i (\chi_1 - \chi_{g_i}). \quad (1.3)$$

Remark 1. 3. Note that χ_Y depends only on the unordered set of conjugacy classes of g_1, \dots, g_r . Suppose $C = \mathbb{P}^1$. Then the existence of a cover $Y \rightarrow C$ with monodromy conjugate to $\{g_1, \dots, g_r\}$ implies that there are elements g'_1, \dots, g'_r and a permutation σ such that g'_i is conjugate to $g_{\sigma(i)}$ and $g'_1 \dots g'_r = 1$. If $\text{char } k$ is prime to $|G|$, the converse is true; see Proposition 1.2 below.

Remark 1. 4. It seems natural to call a character χ of a finite group a *Hurwitz character* if χ is of the form (1.3) for some set of generators

g_1, \dots, g_r with $g_1 \dots g_r = 1$. It follows from Proposition 1.1 and the Riemann Existence theorem that a Hurwitz character is the character of a rational representation (not merely a virtual rational representation) of G ; a purely group-theoretic proof of this fact is given by Scott in [25]. We call such a representation a *Hurwitz representation* of G .

The question of which characters of a finite group are Hurwitz characters is purely combinatorial, and seems quite mysterious; we will discuss this question further in section 5.

We should also note that χ_Y , while strictly speaking an ℓ -adic character, takes values in \mathbb{Q} ; we will freely refer to the values of χ_Y as rational numbers in what follows.

Proof. The proposition follows from the Lefschetz fixed point formula in étale cohomology; see for instance [21, V, Cor. 2.8]. The formula given there for χ_Y is

$$\chi_Y = 2\chi_{triv} + 2(g(C) - 1)\chi_1 + \sum_i a_{p_i},$$

where a_{p_i} is the Artin character associated to the branch point $p_i \in C(k)$. It thus suffices to show that

$$a_{p_i} = \chi_1 - \chi_{g_i}.$$

Let y_1, \dots, y_m be the points of $Y(k)$ lying over p_i . Take $g \in G$ with $g \neq 1$. Then $-(\chi_1(g) - \chi_{g_i}(g))$ is the number of the y_j which are fixed by g . On the other hand,

$$a_{p_i}(g) = -\sum_j i_j(g)$$

where $i_j(g)$ is the multiplicity of the fixed point of g at y_j . Since $\text{char } k$ is prime to the order of g_i , the map $Y \rightarrow C$ is tamely ramified at y , so $i_j(g) = 1$ for each y_j which is a fixed point. ([21, p. 188].) We have thus shown that $a_{p_i}(g) = \chi_1(g) - \chi_{g_i}(g)$ for all $g \neq 1$; since a_{p_i} and $\chi_1 - \chi_{g_i}$ are both orthogonal to the trivial character, they are equal. ■

We recall for future reference Grothendieck's computation of the prime-to- p fundamental group of a curve, which we use here as a replacement for the Riemann existence theorem in characteristic p .

PROPOSITION 1.2. *Let k be an algebraically closed field. Let G be a finite group whose order is prime to $\text{char } k$, or an arbitrary finite group if $\text{char } k = 0$. If g_1, \dots, g_r are generators for G satisfying $g_1 \dots g_r = 1$,*

there exists an Galois covering $Y \rightarrow \mathbb{P}^1$ with Galois group G , branched at r points with monodromy g_1, \dots, g_r .

Proof. Immediate from [11, XIII,2.12]. \blacksquare

We will give examples in the following sections to show that judicious choices of $\{G, H, g_1, \dots, g_r\}$ yield interesting endomorphism algebras of Hecke type.

2. METACYCLIC GROUPS

Let F be a totally real number field. In this section, we use covers of \mathbb{P}^1 with metacyclic Galois groups in order to produce curves with real multiplication by certain totally real number fields.

Let p be a prime, and n be an integer such that $n|p-1$. Let $G = G_{p,n}$ be the metacyclic group

$$\langle \sigma, \alpha : \sigma^p = \alpha^n = 1, \alpha\sigma\alpha^{-1} = \sigma^k \rangle$$

where k is an element of order n in $(\mathbb{Z}/p\mathbb{Z})^*$. Let H be the subgroup of G generated by α . Let ℓ be a prime not equal to $\text{char } k$. For consistency with étale cohomology, we will take $\bar{\mathbb{Q}}_\ell$ as the algebraically closed characteristic 0 base field in which we compute the irreducible representations of G .

The irreducible $\bar{\mathbb{Q}}_\ell$ -representations of G fall into two types:

- n one-dimensional representations V_a , indexed (non-canonically) by $a \in \mathbb{Z}/n\mathbb{Z}$; these arise as compositions

$$G \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \bar{\mathbb{Q}}_\ell^*$$

where the second map sends 1 to ω_n^a , for some fixed n th root of unity $\omega_n \in \bar{\mathbb{Q}}_\ell$.

- $(p-1)/n$ representations of dimension n , which are induced from $\langle \sigma \rangle$.

We denote by W the direct sum of all the n -dimensional irreducible $\bar{\mathbb{Q}}_\ell$ -representations of G .

Let g_1, \dots, g_r be non-trivial elements of G , and for each i in $1, \dots, r$ let d_i be either 0 (if g_i has order p) or $n/\text{ord}(g_i)$ (if g_i has order dividing n .) Note that d_i determines the cyclic group generated by g_i up to conjugacy. Recall that, for any $g \in G$, we denote by χ_g the character of G induced from the trivial character on $\langle g \rangle$. We can compute

$$\chi_{g_i} = d_i \chi(W) + \sum_{a \in \mathbb{Z}/n\mathbb{Z}: ad_i=0} \chi(V_a).$$

Suppose now that k is an algebraically closed field of characteristic prime to $|G|$. Let Y be a Galois cover of \mathbb{P}^1 , with Galois group G , branched at r points with monodromy g_1, \dots, g_r . It follows from Proposition 1.1 that

$$\chi_Y = m\chi(W) + \sum_{a \in \mathbb{Z}/n\mathbb{Z}} c_a \chi(V_a)$$

where

$$c_a = \begin{cases} -2 + |\{i : ad_i \neq 0\}| & a \neq 0 \\ 0 & a = 0 \end{cases}$$

and

$$m = -2n + \sum_i (n - d_i).$$

Let X be the quotient curve Y/H . Since no non-trivial element of V_a is fixed by H for $a \neq 0$, we have

$$H_1(X, \bar{\mathbb{Q}}_\ell) \cong (W^H)^{\oplus m}$$

as $\bar{\mathbb{Q}}_\ell[H \backslash G/H]$ -modules.

Let $W_{\mathbb{Q}}$ be the (unique) \mathbb{Q} -representation of G such that $W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell = W$. Then the map

$$\mathbb{Q}[H \backslash G/H] \rightarrow \text{End}(H^1(X, \bar{\mathbb{Q}}_\ell)^H) = \text{End}((W^H)^{\oplus m})$$

factors through

$$p : \mathbb{Q}[H \backslash G/H] \rightarrow \text{End}(W_{\mathbb{Q}}^H).$$

An element of $\mathbb{Q}[H \backslash G/H]$ acts as the zero endomorphism of $\text{Jac}(X)$ if and only if it acts as zero on $H^1(X, \bar{\mathbb{Q}}_\ell)^H$, which is to say if and only if it is the kernel of p . We conclude that $\mathcal{H}_{X/\mathbb{P}^1}$ is isomorphic to the image of p . Now the image of $\mathbb{Q}[G]$ in $\text{End}(W_{\mathbb{Q}})$ is isomorphic to the $n \times n$ matrix algebra over $\mathbb{Q}(\zeta_p^{(n)})$, and π_H maps to a projection of rank 1. It follows that the image of $\mathbb{Q}[H \backslash G/H] = \mathbb{Q}[\pi_H G \pi_H]$ in $\text{End}(W_{\mathbb{Q}}^H)$ is isomorphic to $\mathbb{Q}(\zeta_p^{(n)})$.

Note that the dimension of W^H is $(p-1)/n$. So if n is even and $m = 2$, we have exhibited a real field of degree $(p-1)/n$ acting on $\text{Jac}(X)$, which is of dimension $(p-1)/n$. So X has real multiplication.

There are only finitely many possibilities for n and d_1, \dots, d_r satisfying $m = 2$. Of these, only nine possibilities can arise from $G_{p,n}$ by means of the above construction:

- 1a. $n = 2; \{d_i\} = \{0, 0, 1, 1\}$
- 1b. $n = 2; \{d_i\} = \{0, 1, 1, 1, 1\}$
- 1c. $n = 2; \{d_i\} = \{1, 1, 1, 1, 1, 1\}$
- 2a. $n = 4; \{d_i\} = \{0, 1, 1\}$
- 2b. $n = 4; \{d_i\} = \{1, 1, 2, 2\}$
- 3a. $n = 6; \{d_i\} = \{2, 2, 3, 3\}$
- 3b. $n = 6; \{d_i\} = \{1, 1, 2\}$
- 4. $n = 8; \{d_i\} = \{1, 1, 4\}$
- 5. $n = 10; \{d_i\} = \{1, 2, 5\}$.

The first three cases give examples of curves whose Jacobians have real multiplication by real cyclotomic fields $\mathbb{Q}(\zeta_p^+)$. The families of curves described by Tautz, Top, and Verberkmoes [27] fall under cases 1a and 1b above. In case 1c, the curve Y is a cover of \mathbb{P}^1 whose Galois group is the dihedral group D_{2p} . The subcover corresponding to $\mathbb{Z}/p\mathbb{Z} \subset D_{2p}$ is a double cover of \mathbb{P}^1 branched over six points—that is, a genus 2 curve C' . The curves X in case 1c are exactly those produced by Brumer [2] as quotients of étale covers of genus 2 curves. The curves produced by Mestre in [19] are special cases of case 1c, arising when C' admits a non-constant morphism to an elliptic curve.

In cases 2b and 3a, the argument above results in the following two propositions.

PROPOSITION 2.1. *Let g_2, g'_2, g_4, g'_4 be elements of $G_{p,4}$, of orders 2, 2, 4, 4 respectively, satisfying $g_2 g'_2 g_4 g'_4 = 1$. Let H be an order-4 subgroup of $G_{p,4}$. Let k be an algebraically closed field of characteristic greater than 2.*

Suppose Y/k is a Galois cover of \mathbb{P}^1/k , with Galois group $G_{p,4}$, branched at four points with monodromy g_2, g'_2, g_4, g'_4 , and let $X = Y/H$.

Then X has real multiplication by the index-4 subfield of $\mathbb{Q}(\zeta_p)$.

PROPOSITION 2.2. *Let g_2, g'_2, g_3, g'_3 be elements of $G_{p,6}$, of orders 2, 2, 3, 3 respectively, satisfying $g_2 g'_2 g_3 g'_3 = 1$. Let H be an order-6 subgroup of $G_{p,6}$. Let k be an algebraically closed field of characteristic greater than 3.*

Suppose Y/k is a Galois cover of \mathbb{P}^1/k , with Galois group $G_{p,6}$, branched at four points with monodromy g_2, g'_2, g_3, g'_3 , and let $X = Y/H$.

Then X has real multiplication by the index-6 subfield of $\mathbb{Q}(\zeta_p)$.

Case 2a provides further examples of a curve with real multiplication by $\mathbb{Q}(\zeta_p^{(4)})$; likewise, case 3b provides further examples of real multiplication by $\mathbb{Q}(\zeta_p^{(6)})$. Cases 4 and 5 bring new fields of real multiplication into the picture; we record the results in the following propositions.

PROPOSITION 2.3. *Let g_1, g_2, g_3 be elements of $G_{p,8}$, of orders 2, 8, 8 respectively, satisfying $g_1 g_2 g_3 = 1$. Let H be an order-8 subgroup of $G_{p,8}$. Let k be an algebraically closed field of characteristic greater than 2.*

Suppose Y/k is a Galois cover of \mathbb{P}^1/k , with Galois group $G_{p,8}$, branched at three points with monodromy g_1, g_2, g_3 , and let $X = Y/H$.

Then X has real multiplication by the index-8 subfield of $\mathbb{Q}(\zeta_p)$.

PROPOSITION 2.4. *Let g_1, g_2, g_3 be elements of $G_{p,10}$, of orders 2, 5, 10 respectively, satisfying $g_1 g_2 g_3 = 1$. Let H be an order-10 subgroup of $G_{p,10}$. Let k be an algebraically closed field of characteristic prime to 10.*

Suppose Y/k is a Galois cover of \mathbb{P}^1/k , with Galois group $G_{p,10}$, branched at three points with monodromy g_1, g_2, g_3 , and let $X = Y/H$.

Then X has real multiplication by the index-10 subfield of $\mathbb{Q}(\zeta_p)$.

We note in the following Proposition that the covers described above actually do exist.

PROPOSITION 2.5. *Suppose $\text{char } k$ does not divide $2p$ (resp. $6p, 2p, 10p$). Then there exists a cover Y of \mathbb{P}^1/k with the branching properties described in Proposition 2.1 (resp. 2.2, 2.3, 2.4).*

Proof. Immediate from Proposition 1.2. ■

Note that we have now completed the proof of cases 4 and 5 of the Main Theorem.

3. MORE DIHEDRAL GROUPS

In this section, we produce some curves whose Jacobians have real multiplication using dihedral groups of order $2m$, where m is odd but not necessarily prime. Let $G = D_m$ be the dihedral group of order $2m$:

$$\langle \sigma, \tau : \sigma^m = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle$$

Let H be the subgroup generated by τ .

The group G has two irreducible \mathbb{Q}_ℓ -representations of dimension 1, and $(m-1)/2$ of dimension 2. Denote by V_1 and V_{-1} the representations of dimension 1, and by W the direct sum of the 2-dimensional representations. Note that W splits over \mathbb{Q} into a direct sum of representations

$$W = \bigoplus_{d|m: d \neq 1} W_d$$

where $\dim W_d = \phi(d)$, and W_d is the base change to $\bar{\mathbb{Q}}_\ell$ of an irreducible \mathbb{Q} -representation $W_{d;\mathbb{Q}}$. Now the image of $\mathbb{Q}[G]$ on $\text{End}(W_{d;\mathbb{Q}})$ is a 2×2 matrix algebra over $\mathbb{Q}(\zeta_d^+)$, and π_H is sent to a rank 1 projection. So the image of $\mathbb{Q}[H \backslash G / H] = \mathbb{Q}[\pi_H G \pi_H]$ in $\text{End}(W_{d;\mathbb{Q}})$ is isomorphic to $\mathbb{Q}(\zeta_d^+)$.

Suppose $m = pq$, where p and q are distinct odd primes. Let g_1, g_2, g_3, g_4 be elements of G conjugate to $\tau, \tau, \sigma^p, \sigma^q$ respectively, and satisfying

$$g_1 g_2 g_3 g_4 = 1.$$

(For instance, we can take $g_1 = \tau, g_2 = \tau \sigma^{-q-p}, g_3 = \sigma^p, g_4 = \sigma^q$.) Let k be an algebraically closed field of characteristic prime to $|G|$, and let Y be a Galois cover of \mathbb{P}^1 , branched at 4 points with monodromy g_1, g_2, g_3, g_4 . It then follows from Proposition 1.1 that

$$(\rho_Y)^H = (W_{pq}^H)^{\oplus 2}.$$

Let X be the quotient curve Y/H . Then X is a curve of genus $(1/2)(p-1)(q-1)$. By the same argument as the one preceding Proposition 2.1, the image of $\mathbb{Q}[H \backslash G / H]$ in $\text{End}(\text{Jac}(X))$ is isomorphic to its image in $\text{End}(W_{pq;\mathbb{Q}})$, which is $\mathbb{Q}(\zeta_{pq}^+)$. So X has real multiplication by $\mathbb{Q}(\zeta_{pq}^+)$.

Now consider the case $m = p^\alpha$, with p an odd prime and $\alpha > 1$. Let g_1, g_2, g_3, g_4, g_5 be elements of G conjugate to $\tau, \tau, \tau, \tau, \sigma^{p^{\alpha-1}}$ respectively, and satisfying

$$g_1 g_2 g_3 g_4 g_5 = 1.$$

For instance, we may take $g_1 = g_2 = g_3 = \tau, g_4 = \tau \sigma^{-p^{\alpha-1}}, g_5 = \sigma^{p^{\alpha-1}}$. Let k be an algebraically closed field of characteristic prime to $|G|$, and let Y be a Galois cover of \mathbb{P}^1 , branched at 5 points with monodromy g_1, g_2, g_3, g_4, g_5 .

Another application of Proposition 1.1 gives

$$(\rho_Y)^H = (W_{p^\alpha}^H)^{\oplus 2}.$$

Again setting $X = Y/H$, we have X a curve of genus $(1/2)(p^\alpha - p^{\alpha-1})$ whose Jacobian has real multiplication by $\mathbb{Q}(\zeta_{p^\alpha}^+)$.

One can check that the dihedral groups of orders $2p, 2pq$, and $2p^\alpha$ are the only ones which produce curves with real multiplication via the above strategy.

4. FAMILIES OF JACOBIANS WITH REAL MULTIPLICATION

In the sections above, we have exhibited curves with real multiplication by various totally real abelian number fields. These curves arose as branched covers $X \rightarrow \mathbb{P}^1$ with specified ramification above the branch points. By moving the branch points, we naturally expect to obtain continuously varying families of curves X with real multiplication.

The problem is that, *a priori*, we do not know that moving the branch points alters the curve. For instance, it might be the case that there is a single curve X such that *every* 4-branched cover of \mathbb{P}^1 with Galois group $G_{p,4}$ and monodromy elements of order 2, 2, 4, 4 is isomorphic to X ! Our goal in this section is to prove that such annoying circumstances do not, in fact, arise.

For the length of this section, we fix:

- a finite group G with a subgroup H ;
- an algebraically closed field K with characteristic prime to $|G|$;
- an r -branched Galois cover $Y/K \rightarrow \mathbb{P}^1/K$ with Galois group G and monodromy elements g_1, \dots, g_r .

Let χ_Y the character of G acting on $H^1(Y, \bar{\mathbb{Q}}_\ell)$, as computed in Proposition 1.1.

We let $X = Y/H$ be the quotient curve, and $\pi : X \rightarrow \mathbb{P}^1$ the projection map. In $X \times X$ we have a divisor $D = X \times_{\mathbb{P}^1} X$; concretely, the closed points of D are the points $(x_1, x_2) : \pi(x_1) = \pi(x_2)$. The irreducible components of D are in bijection with the double cosets HgH of H in G . Write D_g for the component of D corresponding to HgH . An important step in the argument below is the observation that, in most of the cases we consider, D_g has negative self-intersection. The self-intersection is computed by means of the following proposition.

PROPOSITION 4.1. *The self-intersection of D_g is*

$$|H|^{-2} \sum_{g_1, g_2 \in HgH} 2 - \chi_Y(g_1 g_2^{-1}).$$

Remark 4. 1. The summand $2 - \chi_Y(g_1 g_2^{-1})$ can be thought of as the number of fixed points of the action of $g_1 g_2^{-1}$ on Y , whenever $g_1 \neq g_2$.

Proof. Let \tilde{D}_g be the graph of g in $Y \times Y$. It follows from the Lefschetz fixed point theorem in étale cohomology [21, VI,12.3] that

$$\tilde{D}_{g_1} \cdot \tilde{D}_{g_2} = \tilde{D}_{g_1 g_2^{-1}} \cdot \tilde{D}_1 = \sum_{i=0}^2 (-1)^i \text{Tr}(g_1 g_2^{-1} | H^i(Y, \bar{\mathbb{Q}}_\ell)) = 2 - \chi_Y(g_1 g_2^{-1}).$$

Moreover, if $f : Y \rightarrow X$ is the quotient map, we have

$$(f \times f)^* D_g = \bigcup_{g_1 \in HgH} \tilde{D}_{g_1}.$$

The result now follows. \blacksquare

Suppose now that K has an algebraically closed subfield k , such that K has finite transcendence degree over k ; we may think of k as a field of constants and K as the algebraic closure of a function field over k . In particular, we will say that the branched cover $\pi : X/K \rightarrow \mathbb{P}_K^1$ is *non-varying* if it fits into a diagram

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \longrightarrow & X_0 \\ \pi \downarrow & & \downarrow & & \downarrow \pi_0 \\ \mathbb{P}_K^1 & \xrightarrow{\alpha} & \mathbb{P}_K^1 & \xrightarrow{g} & \mathbb{P}_k^1 \end{array}$$

where the right-hand square is Cartesian, g is the base change of the structure map $\text{Spec } K \rightarrow \text{Spec } k$, and α is some automorphism of \mathbb{P}_K^1 .

We show below that, under a certain numerical condition on the ramification of π , that “a branched cover from a curve defined over k to \mathbb{P}^1 is non-varying.”

THEOREM 4.1. *Suppose $X = X_0 \times_k K$ for some curve X_0/k . Suppose furthermore that there exists a double coset HgH with cardinality $|H|^2$ such that*

$$\sum_{g_1, g_2 \in HgH} 2 - \chi_Y(g_1 g_2^{-1}) < 0. \quad (4.4)$$

Then $\pi : X \rightarrow \mathbb{P}_K^1$ is non-varying.

Proof. The basic algebro-geometric ingredient of the proof is the fact that divisors with negative self-intersection cannot vary continuously. More precisely, we need the following lemma.

LEMMA 4.1. *Let S/k be a smooth projective algebraic surface over an algebraically closed field. Let K/k be an algebraically closed extension of k of finite transcendence degree. Suppose that $C \hookrightarrow S_K = S \times_k K$ is an irreducible closed curve with negative self-intersection. Then $C = C_0 \times_k K$ for some irreducible closed curve $C_0 \hookrightarrow S$.*

Proof. Let U be a variety over k , the algebraic closure of whose function field is K . Let C_U be a closed subscheme of $S \times_k U$ whose geometric generic fiber is C . Replacing U by an open subscheme of U , we may assume that U is affine and that the fibers of $C_U \rightarrow U$ are irreducible curves.

Now let P be a k -point of U . Write C_0 for the restriction of C to P ; define $C' = C_0 \times_P K$ and $C'_U = C_0 \times_P U$. Note that C' , being smooth and connected, is an irreducible curve in S_K . We claim that $C = C'$.

To prove the claim, observe that the intersection product can be expressed in terms of Euler characteristics [22, Lecture 12]:

$$(C_0 \cdot C_0) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C_0)) - \chi(\mathcal{O}_S(-C_0)) + \chi(\mathcal{O}_S(-2C_0))$$

and

$$(C \cdot C') = \chi(\mathcal{O}_{S_K}) - \chi(\mathcal{O}_{S_K}(-C)) - \chi(\mathcal{O}_{S_K}(-C')) + \chi(\mathcal{O}_S(-C - C')).$$

Recalling that S is projective, we may think of $\mathcal{O}_S, \mathcal{O}_S(-C_U), \mathcal{O}_S(-C'_U)$, and $\mathcal{O}_S(-C_U - C'_U)$ as coherent sheaves on \mathbb{P}_k^N . Now the fact that Hilbert polynomials of coherent sheaves are locally constant in flat families implies that

$$(C \cdot C') = (C_0 \cdot C_0) = (C \cdot C) < 0.$$

So C_K and C'_K must share an irreducible component; since both divisors are irreducible curves, they must coincide. \blacksquare

We first consider the case in which H is trivial—that is, $X = Y$ is a Galois cover of \mathbb{P}_K^1 . Note that in this case the numerical condition (4.4) says precisely that $g(X) \geq 2$.

Let C be the divisor $X \times_{\mathbb{P}^1} X \hookrightarrow X \times_K X$. Then every component of C is a translate of the diagonal by an automorphism of $X \times_K X$; since $g(X) \geq 2$, every component of C has negative self-intersection. From Lemma 4.1 we have that C is the base change of a divisor $C_0 \subset X_0 \times X_0$.

Now the intersection of C_0 with a fiber $x \times X_0$ is a set S_x of $n = |G|$ points in X_0 (counted with multiplicity). One can thus define a morphism

$$\eta_0 : X_0 \rightarrow \mathrm{Sym}^n X_0$$

which sends x to S_x . Let F_0 be the image of η_0 , and write F, η for $F_0 \times_k K, \eta_0 \times_k K$. Now $\eta(x) = \eta(y)$ if and only if $\pi(x) = \pi(y)$, which is to say that η factors as

$$X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\alpha} F,$$

where α is an isomorphism. Then the diagram

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \longrightarrow & X_0 \\ \pi \downarrow & & \eta \downarrow & & \downarrow \eta_0 \\ \mathbb{P}_K^1 & \xrightarrow{\alpha} & F & \xrightarrow{g} & F_0 \end{array}$$

yields the desired result.

We now return to the general case. Let $f_g : \tilde{D}_g \rightarrow D_g$ be the restriction to \tilde{D}_g of the map $f \times f : Y \times Y \rightarrow X \times X$. We claim that f_g is generically one-to-one. It suffices to consider closed points. Suppose

$$(f(y), f(g(y))) = (f(y'), f(g(y'))).$$

Then $y' = h_1(y)$ and $g(y') = h_2(g(y))$ for some $h_1, h_2 \in H$. So $g(h_1(y)) = h_2(g(y))$.

Suppose y (whence also y') is not fixed by any nontrivial element of G . Then $h_2^{-1}gh_1 = g$. Since HgH has cardinality $|H|^2$, this implies that $h_1 = h_2 = 1$, so $y' = y$. We have shown that f_g is one-to-one on closed points away from a closed subset of \tilde{D}_g ; it follows that f_g is generically one-to-one, as claimed. In particular, since \tilde{D}_g is isomorphic to the smooth curve Y , we have that Y is a nonsingular model for the curve D_g .

It follows from Proposition 4.1 that D_g has negative self-intersection. Applying Lemma 4.1 with $S = X_0 \times X_0$ and $C = D_g$, we find that $D_g = D_0 \times_k K$ for some curve $D_0 \in X_0 \times_k X_0$. Since Y is the normalization of D_g , we also have $Y = Y_0 \times k$. Write $p = \pi \circ f : Y \rightarrow \mathbb{P}_K^1$. Then p is a Galois cover. Observe that $2 - \chi_Y(g)$ is a rational number greater than $2 - 2g(Y)$ for all $g \in G$; so (4.4) implies that $g(Y) > 2$. It now follows from the $H = \{1\}$ case treated above that p is non-varying. After composing p with an automorphism of \mathbb{P}_K^1 , we may therefore assume that p is the base change of a Galois cover $p_0 : Y_0 \rightarrow \mathbb{P}_k^1$. This means that the action of G on Y is the base change of an action of G on Y_0 . Thus, p_0 factors through a map $\pi_0 : Y_0/H \rightarrow \mathbb{P}_k^1$, and π is evidently isomorphic to the base change of π_0 . This yields the desired result. \blacksquare

We will now prove that the theorems of the previous sections actually produce infinite families of curves with real multiplication. Our main tool will be Theorem 4.1; however, in the cases where the field of real multiplication is $\mathbb{Q}(\zeta_p^{(4)})$ or $\mathbb{Q}(\zeta_p^{(6)})$, we instead use a degeneration argument which yields a slightly stronger result than the application of Theorem 4.1 would.

Suppose K/k is an extension of algebraically closed fields of transcendence degree n . For each $g \geq 2$, let \mathcal{M}_g be the moduli space of smooth

genus g curves. We say a genus g curve X/K is an n -dimensional family of curves over k if the map $\text{Spec } K \hookrightarrow \mathcal{M}_g(K)$ induced by X does not factor through any $\text{Spec } L$, where L is an algebraically closed subextension of K/k of transcendence degree less than n over k .

COROLLARY 4.1. *Let k be an algebraically closed field. Then*

1. *If $p > 5$ is a prime, and $\text{char } k$ does not divide $2p$, there exists a 3-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_p^+)$.*
2. *If p is a prime congruent to 1 mod 4, and $\text{char } k$ does not divide $2p$, there exists a 1-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_p^{(4)})$.*
3. *If p is a prime congruent to 1 mod 6, and $\text{char } k$ does not divide $6p$, there exists a 1-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_p^{(6)})$.*
6. *If p, q are distinct odd primes, and $\text{char } k$ does not divide $2pq$, there exists a 1-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_{pq}^+)$.*
7. *If p is an odd prime, and $\text{char } k$ does not divide $2p$, and $\alpha > 1$ is an integer, there exists a 2-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_{p^\alpha}^+)$.*

Proof. We will begin by treating cases 1, 6, and 7 above, using Theorem 4.1. Cases 2 and 3 require us to make an argument on the boundary of the moduli space of curves, which we carry out in the second section of the proof.

Let K_r be the algebraic closure of the function field of the moduli space $\mathcal{M}_{0,r}/k$ of genus 0 curves with r marked points, where $r \geq 4$. Then K_r has transcendence degree $r - 3$ over k . Moreover, the inclusion

$$\bar{\iota} : \text{Spec } K_r \rightarrow \mathcal{M}_{0,r}$$

gives us a set of r points p_1, \dots, p_r on $\mathbb{P}_{K_r}^1$. Suppose $\pi : X \rightarrow \mathbb{P}_{K_r}^1$ is a cover branched at p_1, \dots, p_r . Let $p : Y \rightarrow \mathbb{P}_{K_r}^1$ be the Galois closure of π . As usual, let G be the Galois group of p and H the subgroup corresponding to X via the Galois correspondence. Suppose that G, H, χ_Y satisfy the numerical condition (4.4) of Theorem 4.1. Then X is an $(r-3)$ -dimensional family of curves over k . For suppose otherwise; then X is isomorphic to $X_0 \times_L K_r$, where L/k is an algebraically closed subextension of K_r/k with transcendence degree less than $r - 3$. It then follows from Theorem 4.1 that π is non-varying, which is to say that it is isomorphic the base change to K_r of a cover $\pi_0 : X_0 \rightarrow \mathbb{P}_L^1$. This means, in turn, that the branch locus

of π (i.e. the set of points $\{p_0, \dots, p_r\}$) is the base change to K_r of the branch locus of π_0 . That is, the map

$$\bar{\iota} : \text{Spec } K_r \rightarrow \mathcal{M}_{0,r}$$

factors through $\text{Spec } L$, which contradicts the hypothesis that $\bar{\iota}$ is a geometric generic point of $\mathcal{M}_{0,r}$.

It follows that, in each case of the corollary, we only need to demonstrate that there exists a branched cover $\pi : X \rightarrow \mathbb{P}_{K_r}^1$ branched at p_1, \dots, p_r which has the desired real multiplication and which satisfies (4.4).

We will now work case by case. Let F be the field of real multiplication.

Case 1: $F = \mathbb{Q}(\zeta_p^+)$. We use the branched cover described in section 2 with real multiplication by F . (Recall that this is the cover described by Brumer [2].) Here $r = 6$, the Galois group G is the dihedral group $G_{p,2}$, the subgroup H is generated by an involution τ , and the monodromy at each branch point is conjugate to τ . It follows from Proposition 1.1 that

$$\chi_Y = 2\chi_{triv} - 2\chi_1 + 6(\chi_1 - \chi_\tau) = 2\chi_{triv} + 4\chi_1 - 6\chi_\tau.$$

So

- $\chi_Y(1) = 2p + 2$;
- $\chi_Y(\tau) = -4$;
- $\chi_Y(\sigma^a) = 2$ for all a prime to p .

The double coset $H\sigma H$ has cardinality 4. As g_1, g_2 run through $H\sigma H$, the quotient $g_1g_2^{-1}$ is trivial in 4 cases, conjugate to τ in 8 cases, and conjugate to a nontrivial power of σ in 4 cases. So we have

$$\sum_{g_1, g_2 \in H\sigma H} 2 - \chi_Y(g_1g_2^{-1}) = 4(-2p) + 8 \cdot 6 = 48 - 8p,$$

which is negative once $p > 5$. Note that the assertion of Corollary 4.1 is in fact false for $p = 5$, since the moduli space of abelian surfaces with real multiplication by $\mathbb{Q}(\sqrt{5})$ is only 2-dimensional.

Case 4: $F = \mathbb{Q}(\zeta_{pq}^+)$. Take $r = 4$ and let X be the branched cover described in Section 3 with real multiplication by F . In this case, Proposition 1.1 shows that

$$\chi_Y = 2\chi_{triv} - 2\chi_1 + 2(\chi_1 - \chi_\tau) + (\chi_1 - \chi_{\sigma^p}) + (\chi_1 - \chi_{\sigma^q}).$$

So

- $\chi_Y(1) = 2(p-1)(q-1)$

- $\chi_Y(\tau) = 0$;
- $\chi_Y(\sigma^a) = 2$ for all a prime to pq .

The sum in (4.4) is therefore

$$4(2 - 2(p - 1)(q - 1)) + 8 \cdot 2 = 24 - 8(p - 1)(q - 1),$$

which is negative for all odd primes p, q .

Case 5: $F = \mathbb{Q}(\zeta_{p^\alpha}^+)$. Take $r = 5$ and let X be the branched cover described in Section 3 with real multiplication by F . Here, Proposition 1.1 tells us that

$$\chi_Y = 2\chi_{triv} - 2\chi_1 + 4(\chi_1 - \chi_\tau) + \chi_1 - \chi_{\sigma^{p^{\alpha-1}}}.$$

So

- $\chi_Y(1) = 2 + 2p^\alpha - 2p^{\alpha-1}$
- $\chi_Y(\tau) = -2$;
- $\chi_Y(\sigma^a) = 2$ for all a prime to p .

It follows that the sum in (4.4) is

$$4(2p^{\alpha-1} - 2p^\alpha) + 8 \cdot 4 = 32 - 8(p - 1)p^{\alpha-1},$$

which is negative for any odd prime p and any $\alpha > 1$.

We now turn our attention to the cases $F = \mathbb{Q}(\zeta_p^{(n)})$, where $n = 4, 6$. The methods above will show the existence of 1-dimensional families of curves with real multiplication by F for all sufficiently large p congruent to 1 mod n . In order to prove the desired result for all such p , we introduce a different approach.

Note that to prove our family of curves is 1-dimensional, it suffices to prove it is non-constant. In turn, to prove the family is non-constant, it suffices to prove that it degenerates to a nodal curve (since the generic member of the family is smooth.) In order to study the degeneration of our family of branched covers, we need to recall some facts about compactifications of Hurwitz moduli spaces.

Let C/k be a connected stable r -pointed curve; that is, X has only nodal singularities, and every rational component of the normalization of X has at least three points lying over either marked or nodal points on X . An *admissible cover* of a stable r -pointed curve C is a finite morphism $Y \rightarrow C$ which is étale over all smooth unmarked points of C , and which displays a certain local behavior over the nodes of C ; the precise definition will not concern us here. (See [14, 3.G] for more information.) If C is a smooth curve, an admissible cover of C is just a smooth r -branched cover in the usual sense. The central fact we need is that the moduli stack of admissible covers is a compactification of the moduli stack of smooth branched covers.

LEMMA 4.2. *Let A be a strictly henselian discrete valuation ring with fraction field K . Let $B \rightarrow A$ be a stable r -pointed curve whose generic fiber B_K is smooth, and let $\pi_K : X_K \rightarrow B_K$ be an admissible cover. Suppose that the Galois closure of π_K has Galois group G with order in A^* . Then there exists a tamely ramified extension A' of A , with fraction field K' , such that $\pi_{K'}$ extends to an admissible cover $\pi : X_{A'} \rightarrow B_{A'}$.*

Proof. Argue as in [20, §3.13]. The only difference is that the hypothesis $|G| \in A^*$ replaces Mochizuki's hypothesis $d! \in A^*$. \blacksquare

We use again the notations introduced in section 2. Let G be either $G_{p,4}$ or $G_{p,6}$, let g be an element of order 4 or 3, respectively, and let F be the field $\mathbb{Q}(\zeta_p^{(4)})$ or $\mathbb{Q}(\zeta_p^{(6)})$ respectively. Let h be an element of G of order 2. We think of G as embedded in the symmetric group S_p by means of the action of G on G/H .

Let K be the algebraic closure of $k(t)$. By Proposition 1.2, there exists a Galois cover $Y \rightarrow \mathbb{P}_K^1$ branched at the points $0, 1, \infty$, and t , with monodromy conjugate to g, g^{-1}, h^{-1}, h respectively. Write $X = Y/H$. This cover has a model $X_U \rightarrow \mathbb{P}_U^1$, where U is an open subscheme of a finite étale cover of $\text{Spec } k[t]$, and $X_U \rightarrow \mathbb{P}_U^1$ is smooth. Note that X has real multiplication by F , by the results of section 2.

Let R be the integral closure of $k[t]$ in the fraction field of U , and let $S = \text{Spec } R$. First of all, by the properness of the moduli space $\overline{\mathcal{M}}_{0,4}$ of 4-pointed stable genus 0 curves, there is a finite cover S' of S and a stable 4-pointed genus 0 curve $B \rightarrow S'$ whose geometric generic fiber is P_K^1 marked at $0, 1, \infty$, and t . Let $U' = S' \times_S U$. Then $X_{U'} \rightarrow B_{U'}$ is a smooth 4-branched cover. It follows from Lemma 4.2 that, possibly after another finite extension of S' , the cover $X_{U'}$ can be extended to an admissible cover $X_{S'} \rightarrow B$. For simplicity of notation, replace S by S' .

Let $s \in S$ be a closed point of S lying over the point $t = 0$ of $\text{Spec } k[t]$. Then B_s is a nodal 4-pointed curve of genus 0; any such curve consists of two rational curves meeting at the node, with two marked points lying on each of the irreducible components. Let X_s be the fiber of X_S over s , so that $\pi_s : X_s \rightarrow B_s$ is an admissible cover. Let C, C' be the two irreducible components of B_s . Then $\pi_s^{-1}(C) \rightarrow C$ is a branched degree p cover (not necessarily connected) of a smooth genus 0 curve, which is étale away from the two marked points on C and the node $C \cap C'$. Call these points m_0, m_t , and n .

Now consider an infinitesimal neighborhood of $s \in S$, which is isomorphic to $\text{Spec } k[[u]]$. Since m_0 is a smooth point of C , the completed local ring of B at m_0 is isomorphic to $\text{Spec } k[[u, v]]$. The marking divisor 0 is a section from S to B , which is locally given by the map of rings $k[[u, v]] \rightarrow k[[u]]$

sending v to 0. Let $\text{Spec } A$ be the scheme completing the Cartesian square

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } k[[u, v]] & \longrightarrow & B. \end{array}$$

Then it follows from Abhyankar's Lemma that

$$A \cong \bigoplus_{i=1}^d k[[u, v]][V]/(V^{n_i} - v)$$

where the n_1, \dots, n_d are the lengths of the orbits of the element g of S_p . Now, restricting $\text{Spec } A$ to the fiber $u = 0$, we see that the monodromy around m_0 in the cover $\pi_C : \pi_s^{-1}(C) \rightarrow C$ is conjugate to g . Similarly, π_C has monodromy conjugate to h around m_t . So the monodromy of π_C around n is $g'h'$, where g' is some conjugate of g^{-1} and h some conjugate of h^{-1} . It is now an easy Riemann-Hurwitz computation that every component of $\pi_s^{-1}(C)$ has genus 0 or 1, and that the same is true of $\pi_s^{-1}(C')$. In particular, the stable model of X_s is not a smooth curve of genus $(p-1)/n$. We conclude that the generically smooth family of curves X has a fiber X_s whose stable model is non-smooth; therefore, X is the desired non-constant family of curves with real multiplication by F . ■

One would like to have explicit equations, as in [27],[19] and [2], for the families of branched covers $\pi : X \rightarrow \mathbb{P}_{K_r}^1$ discussed above. These families of branched covers are parametrized by certain Hurwitz spaces, which are smooth schemes of dimension $r - 3$ over k .

To give a rationally parametrized family of branched covers would be to give a branched cover $X \rightarrow \mathbb{P}_U^1$ with the specified monodromy g_1, \dots, g_r , where U is an open subscheme of a rational variety. Such a cover would produce a map from U to a Hurwitz space. In case $r = 4$, the Hurwitz space is a union of curves, whose genera can be computed by passing to characteristic 0 and using combinatorial methods. A computer calculation shows that in the cases $F = \mathbb{Q}(\zeta_{13}^{(4)}), \mathbb{Q}(\zeta_{13}^{(6)})$, the relevant Hurwitz spaces are already genus 1 curves. Our methods thus cannot produce rationally parametrized families of curves whose Jacobians have real multiplication by $\mathbb{Q}(\zeta_{13}^{(4)})$ and $\mathbb{Q}(\zeta_{13}^{(6)})$, and indeed, we have no reason to expect that such families exist.

The situation is different in the case $F = \mathbb{Q}(\zeta_{pq}^+)$. In fact, one can rather easily construct a rationally parametrized family of curves with real multiplication by F .

PROPOSITION 4.2. *Let p and q be odd primes, let k be an algebraically closed field of characteristic prime to $2pq$, and let X/k be the non-singular curve with affine model*

$$T_{pq}(y/2) = P(x)/2(x-1)^p(x-\lambda^2)^q$$

where T_{pq} is the degree pq Tchebyscheff polynomial satisfying

$$T_{pq}((z+z^{-1})/2) = (z^{pq} + z^{-pq})/2$$

and $P(x)$ is the degree $p+q$ polynomial such that

$$P(w^2) = (w-1)^{2p}(w-\lambda)^{2q} + (w+1)^{2p}(w+\lambda)^{2q}.$$

Then X is a curve of genus $(1/2)(p-1)(q-1)$ with real multiplication by $\mathbb{Q}(\zeta_{pq}^+)$.

Proof. Let Y be the curve with affine model

$$z^{pq} = \frac{(w-1)^p(w-\lambda)^q}{(w+1)^p(w+\lambda)^q}.$$

One easily checks that the map $Y \rightarrow \mathbb{P}^1$ sending (z, w) to w^2 realizes Y as a cover of the line with Galois group D_{pq} , branched over $0, 1, \infty$, and λ^2 with ramification of order $2, q, 2, p$ respectively. The action of D_{pq} on Y is generated by

$$\tau : (z, w) \rightarrow (z^{-1}, -w)$$

and

$$\sigma : (z, w) \rightarrow (\zeta_{pq} z, w).$$

Let H be the subgroup of D_{pq} generated by τ . Then Y/H is evidently isomorphic to X ; it now follows from the arguments of section 3 that X has the desired real multiplication. \blacksquare

Remark 4. 2. By choosing λ in a field k_0 which is not algebraically closed, one can arrange for X to have a model over k_0 ; however, the endomorphisms of $\text{Jac}(X)$ will typically be defined only over an extension of k_0 . The problem of determining fields of definition and fields of moduli for the families of curves constructed here is part of the active program of constructing models of branched covers over non-algebraically closed fields.

The interested reader should consult the papers of Dèbes, Douai, and Em-salem [6],[7].

5. QUESTIONS

The main goal of this paper has been to exhibit smooth curves whose Jacobians had many endomorphisms of Hecke type. The negative version of this question—what endomorphism algebras *cannot* act on the Jacobian of a smooth curve?—seems substantially more difficult. In fact, we do not know of a single example of an algebra E which embeds in the endomorphism algebra of a principally polarized abelian variety but which can be proven not to embed in the endomorphism algebra of the Jacobian of a smooth curve. In this section, we frame some questions in the direction of bounds on endomorphisms of Hecke type.

5.1. A non-Galois version of Hurwitz’s theorem?

Hurwitz showed [16] that a curve of genus $g > 1$ over a field k of characteristic 0 has an automorphism group of size at most $84(g - 1)$. In other words, if $Y/k \rightarrow C/k$ is a Galois cover with Galois group G , the dimension of $\mathbb{Q}[G]$ is bounded linearly in $g(Y)$.

Now let Y, C, G, H, X be defined as in section 1. Let $\mathcal{H}_{X/C}$ be the endomorphism algebra of Hecke type; recall that $\mathcal{H}_{X/C}$ is a homomorphic image of $\mathbb{Q}[H \setminus G/H]$ and a subalgebra of $\text{End}_0(\text{Jac}(X))$. We will always suppose that X/C is a tamely ramified cover, so that Proposition 1.1 is in effect.

Question A: Is there a universal constant γ such that

$$\dim \mathcal{H}_{X/C} \leq \gamma g(X)$$

for all X, C ?

Hurwitz’s theorem implies that the answer to question A is yes, with $\gamma = 84$, if we impose the additional condition that H is trivial.

Question A can be rephrased purely in terms of combinatorial group theory. First of all, if $\pi : X \rightarrow C$ is a branched cover, we may compose π with a map $\psi : C \rightarrow \mathbb{P}^1$. The endomorphism algebra of Hecke type associated to the branched cover $\psi \circ \pi : X \rightarrow \mathbb{P}^1$ is easily seen to contain $\mathcal{H}_{X/C}$. So Question A does not change if we replace C by \mathbb{P}^1 . When $C = \mathbb{P}^1$, the action of G on $H^1(Y, \mathbb{Q}_\ell)$ is described by a certain Hurwitz character (see Remark 1.4) and the dimension of $\mathcal{H}_{X/C}$ can be determined from that character.

To be precise: let χ be a Hurwitz character of a finite group G , and let H be a subgroup of G . We can decompose χ as

$$\chi = \sum_{\psi} a_{\psi}(\chi)\psi$$

where ψ ranges over the irreducible characters of G . If ψ is a character of G (trivial or not), write $\psi(H)$ for the inner product of χ with the character χ_H induced from the trivial character on H . Now define

$$A(G, H, \chi) = \sum_{\psi: a_{\psi}(\chi) > 0} \psi(H)^2.$$

Let $Y \rightarrow \mathbb{P}^1$ be a branched cover with Galois group G giving rise to the Hurwitz character χ on $H^1(Y, \mathbb{Q}_{\ell})$, and let $X = Y/H$. Then $A(G, H, \chi)$ is precisely the dimension of $\mathcal{H}_{X/\mathbb{P}^1}$. Note also that $\chi(H) = 2g(X)$.

We arrive at the following equivalent formulation for Question A.

Question A’: Is there a universal constant γ such that

$$A(G, H, \chi) \leq \gamma\chi(H)$$

for all G, H, χ ?

Results of Aschbacher, Guralnick, Neubauer, and Thompson give partial results in the direction of question A’ in case H is a *maximal* subgroup of G , the intersection of whose conjugates is trivial. Such subgroups are classified by a theorem of Aschbacher and Scott; the results of the four above authors show that, in all but one case of the Aschbacher-Scott classification, $[G : H]$ is bounded above by a linear function in $\chi(H)$ [13, §1]. Plainly, $A(G, H, \chi) \leq [G : H]$, so in these cases Question A’ is answered in the affirmative. For example, if G is solvable and H is subject to the above restrictions, then γ can be taken to be 2^{10} [24]. More results in this direction will appear in [12].

5.2. Decomposable Jacobians

In [9], Ekedahl and Serre ask whether there are curves over \mathbb{C} of every genus, or even of arbitrarily large genus, whose Jacobians are isogenous to products of elliptic curves. Let X/k be a curve over an algebraically closed field of characteristic 0, and let $d(X)$ be the supremum of $\dim(A)$ over all simple isogeny factors A of $\text{Jac}(X)$. Then we can rephrase Ekedahl and Serre’s question to read: what conditions on the genus of X are imposed by the condition $d(X) = 1$? More generally, we might ask:

Question B: Is there an increasing, unbounded function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all smooth genus g curves X/k ,

$$d(X) \geq f(g)?$$

By taking C to be a genus g_0 curve and X to be the maximal elementary 2-abelian étale cover of C , one finds that such an f would have to be of order at most $c \log g$. Serre has proved that, if k is replaced by a *finite* field \mathbb{F} , and if $d(X)$ is the maximal dimension of an \mathbb{F} -simple isogeny factor of $\text{Jac}(X)$ over \mathbb{F} , then the answer to question B is yes. In fact, de Jong has shown [5] that $f(g)$ can be chosen on the order of $(\log \log g)^{1/2}$ in this case. On the other hand, if k is taken to be an algebraically closed field of characteristic p , the existence of supersingular curves of arbitrarily large genus shows that the answer to question B is no.

In some cases, we can use endomorphisms of Hecke type to bound $d(X)$. Let X, Y, G, H, C be defined as in section 1, with $|G|$ prime to $\text{char } k$.

Denote by M the $\mathbb{Q}[H \backslash G / H]$ -module $H^1(X, \mathbb{Q}_\ell)$. Let V_1, \dots, V_m be the irreducible \mathbb{Q} -representations of G . Recall the decomposition

$$\mathbb{Q}[H \backslash G / H] \cong \sum_i M_{d_i}(\Delta_i). \tag{5.5}$$

given in (1.1).

For each i , write $\pi_{i,j}$ for the element

$$e_{jj} \in M_{d_i}(\Delta_i) \subset \mathbb{Q}[H \backslash G / H].$$

Then $\pi_{i,j}$ is an idempotent, and we can write

$$M \cong \sum_{i,j} \pi_{i,j} M.$$

We call the subspace $\pi_{i,j} M$ a *factor* of M . Note that $e(\pi_{i,j}) \text{Jac}(X)$ is an abelian subvariety of $\text{Jac}(X)$ of dimension $(1/2) \dim_{\mathbb{Q}_\ell}(\pi_{i,j} M)$.

Now define

$$d(X/C) = \max_{i,j} (1/2) \dim(\pi_{i,j} M).$$

(Note that the maximization over j is really redundant, since $\pi_{i,j} M$ and $\pi_{i,j'} M$ have the same dimension.)

The decomposition of $\text{Jac}(X)$, up to isogeny, as

$$\text{Jac}(X) \sim \bigoplus_{i,j} e(\pi_{i,j}) \text{Jac}(X)$$

implies that $d(X) \leq d(X/C)$.

A slight modification of $d(X/C)$ is sometimes useful. We may suppose that the first factor in the decomposition (5.5) is a copy of \mathbb{Q} , which we may think of as the quotient of $\mathbb{Q}[H \backslash G/H]$ by its augmentation ideal. So $d_1 = 1, \Delta_1 = \mathbb{Q}$. Now define

$$d'(X/C) = \max_{i>1, j} (1/2) \dim(\pi_{i,j} M).$$

Note that $\pi_{1,1}$ projects $\text{Jac}(X)$ onto its G -invariant part, which is isogenous to $\text{Jac}(C)$. So we have

$$\text{Jac}(X) \sim \text{Jac}(C) \oplus \bigoplus_{i>1, j} e(\pi_{i,j}) \text{Jac}(X),$$

which implies that

$$d(X) \leq \max(d(C), d'(X/C)).$$

Ekedahl and Serre give many examples of curves X with large genus such that $d(X) = 1$. With the exception of some modular curves, their examples, like ours, are produced by analyzing the action of a finite group on the Jacobian of a Galois cover of \mathbb{P}^1 . In our notation, the Ekedahl-Serre examples are given as branched covers of low-genus curves C with $d(C) = 1$, and all have $d'(X/C) = 1$.

We are now motivated to ask whether this strategy has any chance of providing a negative answer to Question B.

Question C: Is there an increasing, unbounded function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all smooth genus g curves X/k and all branched covers $X \rightarrow C$,

$$d(X/C) \geq f(g)?$$

We can rephrase question C as a question in combinatorial group theory. By composing the map $X \rightarrow C$ with an arbitrary map $C \rightarrow \mathbb{P}^1$, we may think of X as a branched cover of the projective line; since $\mathcal{H}_{X/C}$ is contained in $\mathcal{H}_{X/\mathbb{P}^1}$, we may restrict to the case $C = \mathbb{P}^1$.

Let G be a finite group, H a subgroup of G , and V be a Hurwitz representation of G . We decompose V into irreducible \mathbb{Q} -representations as

$$V = \bigoplus_i V_i^{\oplus a_i(V)}.$$

Write χ for the character of V , and χ_i for the character of V_i .

Now V^H is a $\mathbb{Q}[H \backslash G / H]$ -algebra. Then

$$\dim_{\mathbb{Q}} \pi_{i,j} V^H = \begin{cases} a_i(V) \dim_{\mathbb{Q}} \Delta_i = \langle \chi_i, \chi \rangle & d_i > 0 \\ 0 & d_i = 0 \end{cases}$$

Suppose Y/C is a cover of curves with Galois group G such that the action of G on $H^1(Y, \mathbb{Q}_\ell)$ is given by $V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Let $X = Y/H$. Then the dimension of $e(\pi_{i,j})(\text{Jac}(X))$ is $(1/2) \dim_{\mathbb{Q}} \pi_{i,j} V^H$.

This leads us to the following equivalent formulation of question C:

Question C': Let G be a finite group, V a Hurwitz representation of G , and H a subgroup of G . Let Σ be the set of i in $[1..m]$ such that $d_i > 0$.

Is there an increasing, unbounded function $f : \mathbb{N} \rightarrow \mathbb{N}$, independent of G, H , and V , such that

$$\max_{i \in \Sigma} \langle \chi_i, \chi \rangle \geq f(\dim(V^H))?$$

This problem seems rather difficult even for the case of Galois covers, where H is trivial.

We will discuss one special case, from which Ekedahl and Serre derive many examples of curves with decomposable Jacobians ([9, §4]). Let X/\mathbb{P}^1 be a Galois cover with group G , and suppose that G admits a non-trivial homomorphism to $\mathbb{Z}/2\mathbb{Z}$, with kernel K . We thus obtain a diagram

$$X \rightarrow X_0 \rightarrow \mathbb{P}^1,$$

where X_0 is a hyperelliptic curve. Let V be the Hurwitz representation of G corresponding to the cover $X \rightarrow \mathbb{P}^1$, and suppose that

$$\max_i \langle \chi_i, \chi \rangle < B.$$

We will derive an upper bound, in terms of B , for $g(X) = (1/2) \dim V^H$.

Let χ be the character of V , and χ_i the character of V_i . Now each irreducible *complex* character ψ of G is a constituent of a unique χ_i . In particular,

$$\max_{\psi} \langle \psi, \chi \rangle \leq \max_i \langle \chi_i, \chi \rangle < B.$$

Now if θ is an irreducible complex character of K , we have

$$\langle \theta, \chi|_K \rangle_K = \langle \text{Ind}_K^G \theta, \chi \rangle_G.$$

Since $\text{Ind}_K^G \theta$ is a sum of either one or two irreducible characters of G we have that $\langle \theta, \chi|_K \rangle_K < 2B$.

Suppose furthermore that $g(X_0) \geq 2$ and that the cover $X \rightarrow X_0$ is étale, as is the case in Ekedahl and Serre's examples. One has by Proposition 1.1 that

$$\chi|_K = 2\chi_{triv} + 2(g(X_0) - 1)\chi_1. \quad (5.6)$$

(Here, χ_{triv} and χ_1 are the trivial and regular characters of K , not G .) In particular, if θ is a non-trivial irreducible complex character of K , then

$$\langle \theta, \chi|_K \rangle_K = 2(g(X_0) - 1)(\theta(1)) < 2B. \quad (5.7)$$

So K has no irreducible complex representations of dimension greater than B . Hence the irreducible complex representations of G have dimension at most $2B$. A theorem of Isaacs and Passman [17, (12.23)] now tells us that G has an abelian subgroup A of index at most $((2B)!)^2$.

We can also bound the order of an element of G . Let g be an element of G of some order n , and let W be an irreducible complex representation of G in which $\langle g \rangle$ acts faithfully. Since the dimension of W is at most $2B$, the field of definition of the character of W must have degree at least $\phi(n)/2B$ over \mathbb{Q} . In particular, if V_i is the irreducible \mathbb{Q} -representation of G containing W , then $\dim_{\mathbb{Q}} \Delta_i$ is at least $\phi(n)/2B$.

We know from the description of $\chi|_K$ in (5.6) that $\langle \chi_i|_K, \chi|_K \rangle > 0$. Let τ be the character of G obtained by pullback from the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$. Denote $\chi_i \otimes \tau$ by $\chi_{i'}$. Then either χ_i or $\chi_{i'}$ must occur with positive multiplicity in the decomposition of χ ; that is, either a_i or $a_{i'}$ is positive. Note that $\Delta_{i'} = \Delta_i$. We thus have

$$B > \max_i \langle \chi_i, \chi \rangle = \max_i a_i(V) \dim_{\mathbb{Q}} \Delta_i > \phi(n)/2B.$$

We conclude that $\phi(n) < 2B^2$, where n is the order of any element of G . This implies in turn that $n < o(B^{2-\epsilon})$.

Since K is the Galois group of a connected étale cover of X_0 , it can be generated by $2g(X_0)$ elements. Write A' for $A \cap K$. Then $[K : A']$ is at most $((2B)!)^2$, and it follows from the Reidemeister-Schreier theorem that A' can be generated by $2g(X_0)((2B)!)^2 + 1 - ((2B)!)^2$ elements. For simplicity, we say that A' can be generated by $2g(X_0)((2B)!)^2$ elements. Since every element of A' has order at most $o(B^{2-\epsilon})$, we have

$$\log |A'| = O(2g(X_0)((2B)!)^2 \log B).$$

Furthermore, since $g(X_0) = O(B)$ by (5.7), we have

$$\log |A'| = O(B \log B ((2B)!)^2).$$

Since $[G : A']$ is bounded above by a constant times $((2B)!)^2$, we also have $\log |G| = O(B \log B((2B)!)^2)$, whence

$$\log \log |G| = O(2 \log(2B)!) = O(B \log B).$$

Note that the genus of X is just $1 + (g(X_0) - 1)|G|$, which is $O(B)|G|$. So

$$\log \log g(X) = O(B \log B),$$

from which we conclude that

$$d(X/\mathbb{P}^1) \geq c \frac{\log \log g(X)}{\log \log \log g(X)}$$

for some absolute constant c .

ACKNOWLEDGMENT

It is my pleasure to thank Armand Brumer, who generously communicated his ideas to me and thereby inspired this paper. I also thank Columbia University for the hospitality I enjoyed while preparing much of the article. I thank Johan de Jong, Robert Guralnick, Martin Isaacs, Adam Logan, and Stefan Wewers for useful discussions, and the referee for thorough and helpful comments on an earlier draft.

REFERENCES

1. P.R. Bending. *Curves of genus 2 with $\sqrt{2}$ multiplication*. PhD thesis, Oxford, 1998.
2. A. Brumer. Personal communication.
3. K. Coombes and D. Harbater. Hurwitz families and arithmetic Galois groups. *Duke Math. J.*, 52(4):821–839, 1985.
4. A.J. de Jong and R. Noot. Jacobians with complex multiplication. In *Arithmetic algebraic geometry*, number 89 in Progr. Math. Birkhäuser Boston, 1991.
5. A.J. de Jong. A theorem of Serre. Preprint.
6. P. Dèbes and J.-C. Douai. Gerbes and covers. *Comm. Algebra*, 27(2):577–594, 1999.
7. P. Dèbes, J.-C. Douai, and M. Emsalem. Familles de Hurwitz et cohomologie non abélienne. *Ann. Inst. Fourier (Grenoble)* 50(1):113–149, 2000.
8. P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compositio Math.*, 90(1):59–59, 1994.
9. T. Ekedahl and J.-P. Serre. Exemples de courbes algébriques à jacobienne complètement décomposable. *C. R. Acad. Sci. Paris, Série I Math.*, 317:509–513, 1993.
10. M. Fried. Fields of definition of function fields and Hurwitz families–groups as Galois groups. *Comm. Algebra*, 5(1):17–82, 1977.
11. A. Grothendieck. *Revêtements étales et groupe fondamental (SGA I)*. Springer, 1971. Lecture Notes in Math. 224.

12. R. Guralnick. Monodromy groups of coverings of curves. Preprint.
13. R. Guralnick. The genus of a permutation group. In *Groups, combinatorics, and geometry (Durham 1990)*, number 165 in LMS Lecture Note Series, pages 351–363. Cambridge Univ. Press, 1992.
14. J. Harris and I. Morrison. *Moduli of curves*. Graduate Texts in Mathematics, 187. Springer-Verlag, 1998.
15. K.-i Hashimoto and N. Murabayashi. Shimura curves as intersections of Humbert surfaces and defining equations of QM-curves of genus two. *Tohoku Math. J. (2)*, 47(2):271–296, 1995.
16. A. Hurwitz. Über algebraische gebilde mit eindeutigen transformationen in sich. *Math. Ann.*, 41:403–442, 1893.
17. I.M. Isaacs. *Character theory of finite groups*. Dover, 1976.
18. N. Jacobson. *Basic Algebra II*. W. H. Freeman, 1989.
19. J.-F. Mestre. Familles de courbes hyperelliptiques à multiplications réelles. In *Arithmetic algebraic geometry*, number 89 in Progr. Math. Birkhäuser Boston, 1991.
20. S. Mochizuki. The geometry of the compactification of the Hurwitz scheme. *Publ. Res. Inst. Math. Sci.*, 31(3):355–441, 1995.
21. J.S. Milne. *Étale cohomology*. Princeton University Press, 1980.
22. D. Mumford. *Lectures on Curves on an Algebraic Surface*. Princeton University Press, 1966.
23. D. Mumford. *Abelian Varieties*. Tata Institute, 1970.
24. M. Neubauer. On monodromy groups of fixed genus. *J. Algebra*, 153(1): 215–261, 1992.
25. L. Scott. Matrices and cohomology. *Ann. of Math. (2)*, 105:473–492, 1977.
26. I. Shimada. A construction of algebraic curves whose Jacobians have non-trivial endomorphisms. *Comment. Math. Univ. St. Paul.*, 43(1):25–34, 1994.
27. W. Tautz, J. Top, and A. Verberkmoes. Explicit hyperelliptic curves with real multiplication and permutation polynomials. *Canad. J. Math.*, 43(5):1055–1064, 1991.
28. G. van der Geer and F. Oort. Moduli of abelian varieties: a short introduction and survey. In *Moduli of Curves and Abelian Varieties*, C. Faber and E. Looijenga, eds., Vieweg, 1999.
29. S. Wewers. *Construction of Hurwitz spaces*. PhD thesis, Essen, 1998.