On the error term in Duke's estimate for the average special value of L-functions

Jordan S. Ellenberg
Princeton University
ellenber@math.princeton.edu

20 Apr 2005

Abstract

Let $\mathcal{F}$ be an orthonormal basis for weight 2 cusp forms of level N. We show that various weighted averages of special values $L(f \otimes \chi, 1)$ over $f \in \mathcal{F}$ are equal to $4\pi c + O(N^{-1+\epsilon})$, where c is an explicit nonzero constant. A previous result of Duke gives an error term of $O(N^{-1/2} \log N)$.

MSC: 11F67 (11F11)

Introduction

Let N be a positive integer, and let $\mathcal{F}$ be an basis for $S_2(\Gamma_0(N))$ which is orthonormal for the Petersson inner product. Let χ be a Dirichlet character.

In [2], Duke proves the estimate

$$\sum_{f \in \mathcal{F}} a_1(f) L(f \otimes \chi, 1) = 4\pi + O(N^{-1/2} \log N) \quad (1)$$

in case N is prime and χ is unramified at N, using the Petersson formula and the Weil bounds on Kloosterman sums.

In this note, we will sharpen the error term in Duke's estimate to $O(N^{-1+\epsilon})$. At the same time, we observe that his techniques generalize to arbitrary N and χ, and to the situation where a_1 is replaced by an arbitrary a_m.

We have in mind an application to the problem of finding all primitive solutions to the generalized Fermat equation

$$A^4 + B^2 = C^p \quad (2)$$

In [3], we show how to associate to a solution of (2) an elliptic curve over $\mathbb{Q}[i]$ with an isogeny to its Galois conjugate and a non-surjective mod p Galois representation. Such curves are parametrized by rational points on a certain modular curve X; following Mazur's method, we can place strong constraints on $X(\mathbb{Q})$ by exhibiting a quotient of the Jacobian of X with Mordell-Weil rank 0. This problem, in turn, reduces via the theorem of Kolyvagin and Logachev to proving the existence of a new form f on level p^2 or $2p^2$ such that the image of f under a certain Hecke operator has an L-function with non-vanishing special value. We can then derive from Duke's estimate that (2) has no solutions for $p > 2 \cdot 10^5$. Using the sharper estimate derived here, we find in [3] that (2) has no solutions for $p \geq 211$.

The author thanks Emmanuel Kowalski for useful discussions about the topic of this paper, and is very grateful to Nathan Ng for finding an error in an earlier version, and for suggesting several helpful sharpenings of the bounds.

Theorem statements

In this section we state various versions of our estimate. If f is a modular form, we always use $a_m(f)$ to denote the Fourier coefficients of the q-expansion of f:

$$f = \sum_{m=0}^{\infty} a_m(f) q^m.$$

As above, we denote by $\mathcal{F}$ a Petersson-orthonormal basis for $S_2(\Gamma_0(N))$.

Write (a_m, L_χ) for the sum

$$\sum_{f \in \mathcal{F}} a_m(f) L(f \otimes \chi, 1)$$

and let q be the conductor of χ.

We obtain a rather complicated bound for (a_m, L_χ), which we state below.

Theorem 1. *Suppose $N \geq 400$, $N \not| q$, and let σ be a real number with $q^2/2\pi \leq \sigma \leq Nq/\log N$. Then we can write*

$$(a_m, L_\chi) = 4\pi\chi(m) e^{-2\pi m/\sigma N \log N} - E^{(3)} + E_3 - E_2 - E_1 + (a_m, B(\sigma N \log N))$$

where

- $|(a_m, B(\sigma N \log N)| \leq 30(400/399)^3 \exp(2\pi) q^2 m^{3/2} N^{-1/2} d(N) N^{-2\pi\sigma/q^2}$;
- $|E_1| \leq (16/3)\pi^3 m^{3/2} \sigma \log N e^{-N/2\pi m\sigma \log N}$;
- $|E_2| \leq (8/9)\pi^5 \zeta^2(7/2) m^{5/2} \sigma^2 N^{-3/2} \log^2 N$;
- $|E_3| \leq (8/3)\zeta^2(3/2)\pi^3 \sigma m^{3/2} N^{-1/2} \log N d(N) e^{-N/2\pi m\sigma \log N}$;
- $|E^{(3)}| \leq 16\pi^3 m \sum_{c>0, N|c} \min[\frac{2}{\pi}\phi(q) c^{-1} \log c, \frac{1}{6}\sigma N \log N m^{1/2} c^{-3/2} d(c)]$.

Proof. Immediate from Propositions 5,6,7,9,10. □

If q, m are considered as constants, the bound above simplifies considerably.

Corollary 2.

$$(a_m, L_\chi) = 4\pi\chi(m) e^{-2\pi m/\sigma N \log N} + O(N^{-1+\epsilon})$$

where the implied constants depend only on m, q, and ϵ.

Proof. The only thing to check is that the bound on $|E^{(3)}|$ is of order at most $N^{-1+\epsilon}$; one checks this by fixing some cutoff X, say $X = N^3$, and observing that both $\sum_{0<c<X, N|c} c^{-1} \log c$ and $N \log N \sum_{c>X, N|c} c^{-3/2} d(c)$ are $O(N^{-1+\epsilon})$. □

The "true behavior" of (a_m, L_χ) is less clear. One might for instance ask: what is the true asymptotic behavior of $(a_m, L_\chi) - 4\pi\chi(m)$ as N grows with m, q held fixed? More generally, what is the shape of the region in m, q, N-space for which (a_m, L_χ) is close to $4\pi\chi(m)$? One might, for instance, define $f_\delta(N)$ to be the smallest integer such that $|(a_m, L_\chi) - 4\pi\chi(m)| \leq \delta$ for all $m \leq f(N)$. Duke's approach shows that $f_\delta(N) \gg N^{1/2}$, whereas the present results show that $f_\delta(N) \gg N^{3/5}$. (Remark: further expansion of the Bessel function in Taylor series will give $f_\delta(N) \gg N^{1-\epsilon}$, with a constant depending on q, ϵ.) Similarly, one could try to optimize the dependence on q in order to get a result that applied when q is large compared to N.

Proof of the main result

We begin by recalling the Petersson trace formula.

Lemma 3 (Petersson trace formula). *Let m, n be positive integers, and let $\mathcal{F}$ be an orthonormal basis for $S_2(\Gamma_0(N))$.*

Then

$$\frac{1}{4\pi\sqrt{mn}} \sum_{f \in \mathcal{F}} a_m(f) a_n(f) = \delta_{mn} - 2\pi \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c) \tag{3}$$

where $S(m,n;c)$ is the Kloosterman sum for $\Gamma_0(N)$, and J_1 is the J-Bessel function.

Proof. See [4, Th. 3.6]. ☐

We can and do assume that $\mathcal{F}$ consists of eigenforms for T_p for all $p \not| N$, and for w_N.

The Petersson product on $S_2(\Gamma_0(N))$ induces an inner product on the dual space $S_2(\Gamma_0(N))^\vee$. With respect to this product, the left-hand side of (3) is $\frac{1}{4\pi\sqrt{mn}}(a_m, a_n)$.

Lemma 3 immediately gives a bound on the size of (a_m, a_n).

Lemma 4. *We have the bound*

$$|(a_m, a_n) - 4\pi\sqrt{mn}\delta_{mn}| \leq 8\zeta^2(3/2)\pi^2(m,n)^{1/2} mn N^{-3/2} d(N).$$

Proof. Applying the Weil bound

$$|S(m,n;c)| \leq (m,n,c)^{1/2} d(c) c^{1/2}$$

and the fact that $|J_1(x)| \leq x/2$ yields

$$\begin{aligned} |4\pi\sqrt{mn} \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c)| &\leq 4\pi\sqrt{mn} \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-1/2} d(c)(m,n)^{1/2}(2\pi\sqrt{mn}/c) \\ &= 8\pi^2(m,n)^{1/2} mn \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-3/2} d(c). \end{aligned}$$

Now the sum over c is equal to

$$\sum_{b>0} (Nb)^{-3/2} d(Nb)$$

which is bounded above by

$$N^{-3/2}d(N)\sum_{b>0} b^{-3/2}d(b) = \zeta^2(3/2)N^{-3/2}d(N).$$

This yields the desired result. $\square$

Let L_χ be the element of $S_2(\Gamma_0(N))^\vee$ which sends each cusp form f to the special value $L(f\otimes\chi,1)$. Then the value to be estimated is precisely (L_χ, a_m). In order to estimate this product via the Petersson formula, it is necessary to approximate L_χ as a sum of Fourier coefficients. We accomplish this via the standard approximation to $L_\chi(f)$ by a rapidly converging series [5].

We define a linear functional $A(x)$ on $S_2(\Gamma_0(N))$ by the rule

$$A(x)(f) = \sum_{n\geq 1}\chi(n)a_n(f)n^{-1}e^{-2\pi n/x}.$$

Then A is a good approximation to the functional L_χ when x becomes large. Let $B(x) = A(x) - L_\chi$. Let M be an integer such that $f\otimes\chi$ is a cuspform on $\Gamma_1(M)$ for all $f\in\mathcal{F}$.

By the functional equation for $L(f\otimes\chi,s)$, we have

$$B(x)(f) = \sum_{n\geq 1} a_n(w_M(f\otimes\chi))n^{-1}e^{-2\pi nx/M}.$$

When x is on the order of $N\log N$, then $B(x)$ is a short sum, and we want to show it is negligible. The only difficulty is bounding the Fourier coefficients of $w_M(f\otimes\chi)$. This is difficult only in case the conductor of χ has common factors with N, in which case $f\otimes\chi$ is not necessarily an eigenform for any W-operator, even when f is a new form (see [1].)

A crude bound will be enough for us. We define an "average cuspform"

$$g = \sum_{f\in\mathcal{F}} a_m(f)(f\otimes\chi).$$

Then

$$a_n(g) = \chi(n)(a_m, a_n)$$

and it follows from Lemma 4 that

$$|a_n(g)| \leq (8\zeta^2(3/2)\pi^2 m^{3/2}N^{-3/2}d(N))n$$

for all $n\neq m$, while

$$|a_m(g)| \leq 4\pi\sqrt{mn} + (8\zeta^2(3/2)\pi^2 m^{3/2}N^{-3/2}d(N))n$$

when $m = n$.

We have that

$$(a_m, B(x)) = \sum_{f\in\mathcal{F}} a_m(f)\sum_{n>0} a_n(w_M(f\otimes\chi))n^{-1}e^{-2\pi nx/M} = \sum_{n>0} a_n(w_M g)n^{-1}e^{-2\pi nx/M},$$

so it remains to bound the Fourier coefficients of the single form $w_M g$. Write c for the constant $8\zeta^2(3/2)\pi^2 m^{3/2}N^{-3/2}d(N)$.

If τ is a point in the upper half plane, we have

$$\begin{aligned} |g(\tau)| \leq \sum_{n>0} |a_n e^{2\pi i n\tau}| &= \sum_{n>0} |a_n| \exp(-2\pi \operatorname{Im}(n\tau)) \\ &\leq \sum_{n>0} cn \exp(-2\pi \operatorname{Im}(n\tau)) + 4\pi m \exp(-2\pi \operatorname{Im}(m\tau)) \\ &\leq c(2\pi \operatorname{Im}(\tau))^{-2} + 4\pi m. \end{aligned}$$

Choose a positive real constant α. The Fourier coefficient $a_n(w_M g)$ can be expressed as

$$\int_0^1 w_M g(\alpha i + t) \exp(-2\pi i n(\alpha i + t)) dt = \int_0^1 M^{-1}(\alpha i + t)^{-2} g(-1/M(\alpha i + t)) \exp(-2\pi i n(\alpha i + t)) dt. \tag{4}$$

Now $\operatorname{Im}((-1/M(\alpha i + t))) = M^{-1}\alpha|\alpha i + t|^{-2}$. So it follows from (4) that

$$\begin{aligned} |a_n(w_M g)| &\leq \int_0^1 M^{-1}|\alpha i + t|^{-2}[c(2\pi)^{-2}M^2\alpha^{-2}|\alpha i + t|^4 + 4\pi m] \exp(2\pi n\alpha) dt \\ &= cM(2\pi)^{-2} \exp(2\pi i n\alpha)\alpha^{-2} \int_0^1 |\alpha i + t|^2 dt + 4\pi m M^{-1} \exp(2\pi n\alpha) \int_0^1 |\alpha i + t|^{-2} dt \\ &\leq cM(2\pi)^{-2} \exp(2\pi n\alpha)\alpha^{-2}(\alpha^2 + 1) + 4\pi m M^{-1} \exp(2\pi n\alpha)\alpha^{-2}. \end{aligned}$$

Now setting $\alpha = 1/n$ yields

$$|a_n(w_M g)| \leq cM(2\pi)^{-2} \exp(2\pi)(1 + n^2) + 4\pi \exp(2\pi) m M^{-1} n^2.$$

We now use the very rough bound $1 + n^2 \leq n^2(n + 1)$ to obtain

$$\begin{aligned} |(a_m, B(x))| &= |\sum_{n>0} a_n(w_M g) n^{-1} e^{-2\pi n x/M}| \\ &\leq [cM(2\pi)^{-2} \exp(2\pi) + 4\pi m M^{-1} \exp(2\pi)] \sum_{n>0} n(n + 1) e^{-2\pi n x/M} \\ &= \exp(2\pi)(cM(2\pi)^{-2} + 4\pi m M^{-1})(2 \exp(-2\pi x/M))(1 - \exp(-2\pi x/M))^{-3}. \end{aligned}$$

Now M can be taken to be $q^2 N$ where q is the conductor of χ. Let σ be a constant to be fixed later, and set $x = \sigma N \log N$. Finally, suppose $N > 400$ and suppose $\sigma > q^2/2\pi$. First of all, we observe that under the hypothesis on N,

$$\begin{aligned} cM(2\pi)^{-2} + 4\pi m M^{-1} &= 2\zeta^2(3/2) q^2 m^{3/2} N^{-1/2} d(N) + 4\pi m q^{-2} N^{-1} \\ &\leq 15 q^2 m^{3/2} N^{-1/2} d(N). \end{aligned}$$

Also,

$$1 - \exp(-2\pi x/M) = 1 - \exp(-2\pi\sigma \log N/q^2) \leq 1 - 400^{-2\pi\sigma/q^2} \leq 400/399.$$

So, in all, we have proved the following.

Proposition 5. *Suppose $N \geq 400$ and $\sigma > q^2/2\pi$. Then*

$$|(a_m, B(\sigma N \log N)| \leq 30(400/399)^3 \exp(2\pi) q^2 m^{3/2} N^{-1/2} d(N) N^{-2\pi\sigma/q^2}.$$

In other words, we have shown that the error in approximating (a_m, L_χ) by $(a_m, A(x))$ is bounded by a function decreasing quickly in N, if x is chosen on the order of $q^2 N \log N$.

We now turn to the analysis of $(a_m, A(\sigma N \log N))$.

First of all, we have

$$(a_m, A(\sigma N \log N)) = \sum_{f \in \mathcal{F}} a_m(f) \sum_{n>0} \chi(n) a_n(f) n^{-1} e^{-2\pi n/\sigma N \log N} = \sum_{n>0} \chi(n)(a_m, a_n) n^{-1} e^{-2\pi n/\sigma N \log N}$$

which, by Lemma 3, equals

$$4\pi\chi(m) e^{-2\pi m/\sigma N \log N} - 8\pi^2 \sqrt{m} \sum_{n>0} \chi(n) n^{-1/2} e^{-2\pi n/\sigma N \log N} \sum_{\substack{c>0 \\ c \equiv 0 \pmod N}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c).$$

We split the latter sum into two ranges; write

$$E^{(1)} = 8\pi^2 \sqrt{m} \sum_{n>0} \chi(n) n^{-1/2} e^{-2\pi n/\sigma N \log N} \sum_{\substack{c>2\pi\sqrt{mn} \\ c \equiv 0 \pmod N}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c)$$

and

$$E_1 = 8\pi^2 \sqrt{m} \sum_{n>0} \chi(n) n^{-1/2} e^{-2\pi n/\sigma N \log N} \sum_{\substack{0<c\leq 2\pi\sqrt{mn} \\ c \equiv 0 \pmod N}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c).$$

We claim E_1 decreases quickly with N. First, recall that $|J_1(a)| \leq \min(1, a/2)$ for all real a. So

$$|E_1| \leq 8\pi^2 \sqrt{m} \sum_{n>0} n^{-1/2} e^{-2\pi n/\sigma N \log N} \sum_{0<Nb\leq 2\pi\sqrt{mn}} (Nb)^{-1} S(m,n;Nb).$$

Note that the inner sum in $|E_1|$ has nonzero terms only when $n > (N/2\pi\sqrt{m})^2$. In this range, the exponential decay takes over. We observe that $|S(m,n;Nb)| \leq m^{1/2}(Nb)^{1/2} d(Nb) < 2\sqrt{m} Nb$, so we can bound E_1 by

$$\begin{aligned}
|E_1| &\leq 8\pi^2\sqrt{m} \sum_{n>(N/2\pi\sqrt{m})^2} n^{-1/2} e^{-2\pi n/\sigma N \log N} \sum_{0<Nb\leq 2\pi\sqrt{mn}} 2\sqrt{m} \\
&\leq 8\pi^2\sqrt{m} \sum_{n>(N/2\pi\sqrt{m})^2} n^{-1/2} e^{-2\pi n/\sigma N \log N} (2\sqrt{m})(2\pi\sqrt{mn}/N) \\
&= 32\pi^3 N^{-1} m^{3/2} \sum_{n>(N/2\pi\sqrt{m})^2} e^{-2\pi n/\sigma N \log N} \\
&\leq 32\pi^3 N^{-1} m^{3/2} e^{-N/2\pi m\sigma \log N} (1 - e^{-2\pi/\sigma N \log N})^{-1}.
\end{aligned}$$

We now simplify this bound under assumptions on N and σ.

Proposition 6. *Suppose $N \geq 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_1| \leq (16/3)\pi^3 m^{3/2} \sigma \log N e^{-N/2\pi m\sigma \log N}.$$

Proof. This amounts to the observation that $\sigma N \log N \geq 300$, from which it follows that

$$(1 - e^{-2\pi/\sigma N \log N})^{-1} \leq (1/6)\sigma N \log N.$$

□

We now consider the sum $E^{(1)}$ over the range where n is small compared to c. In this range, we use the Taylor approximation

$$|J_1(a) - a/2| \leq (1/16)a^3. \tag{5}$$

So we can write $E^{(1)} = E^{(2)} + E_2$, where

$$E^{(2)} = 8\pi^2\sqrt{m}\sum_{n>0}\chi(n)n^{-1/2}e^{-2\pi n/\sigma N \log N} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} c^{-1}S(m,n;c)(2\pi\sqrt{mn}/c).$$

We claim E_2 decreases with N. For we have by (5) that

$$\begin{aligned} |E_2| &\leq 8\pi^2\sqrt{m}\sum_{n>0} n^{-1/2}e^{-2\pi n/\sigma N \log N} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} c^{-1}S(m,n;c)(1/16)(4\pi\sqrt{mn}/c)^3 \\ &= 32\pi^5 m^2 \sum_{n>0} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} ne^{-2\pi n/\sigma N \log N} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} c^{-4}S(m,n;c). \end{aligned}$$

We now use the Weil bound $S(m,n;c) \leq m^{1/2}c^{1/2}d(c)$ to get

$$\begin{aligned} |E_2| &\leq 32\pi^5 m^{5/2} \sum_{n>0} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} ne^{-2\pi n/\sigma N \log N} c^{-7/2}d(c) \\ &\leq 32\pi^5 m^{5/2} \sum_{n>0}\sum_{b>0} ne^{-2\pi n/\sigma N \log N} N^{-7/2}d(N)b^{-7/2}d(b) \\ &\leq 32\pi^5 m^{5/2} N^{-7/2}d(N)\zeta^2(7/2)\sum_{n>0} ne^{-2\pi n/\sigma N \log N} \end{aligned}$$

So we can write

$$|E_2| \leq 32\pi^5\sqrt{3}\zeta(3)m^{5/2}N^{-7/2}e^{-2\pi/\sigma N \log N}(1 - e^{-2\pi/\sigma N \log N})^{-2}.$$

Proposition 7. *Suppose $N > 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_2| \leq (8/9)\pi^5\zeta^2(7/2)m^{5/2}\sigma^2 N^{-3/2}\log^2 N.$$

Proof. Another use of the bound $(1 - e^{-2\pi/\sigma N \log N})^{-1} \leq (1/6)\sigma N \log N$. □

We now come to $E^{(2)}$, which is the main term of the error

$$|(a_m, L_\chi) - 4\pi\chi(m)e^{-2\pi m/\sigma N \log N}|.$$

Recall from above that

$$E^{(2)} = 16\pi^3 m \sum_{n>0} \sum_{\substack{c>2\pi\sqrt{mn} \\ c=0 \pmod N}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m,n;c).$$

Applying the Weil bound to $S(m,n;c)$ yields the estimate $E^{(2)} = O(N^{-1/2}\log N)$ which appears in [2]. We want to exploit cancellation between the Kloosterman sums in order to improve Duke's bound on $E^{(2)}$.

For simplicity, we carry this out under assumptions on the size of N and σ. For the remainder of this section, assume that

- $N \geq 400$;
- $q^2/2\pi \leq \sigma \leq Nq/\log N$.

Recall that under these hypotheses

$$\sigma N \log N \geq (1/2\pi)400 \log 400 > 300.$$

First of all, we will need a simple bound on the modulus of $1 - e^z$.

Lemma 8. *Let z be a complex number with $|\operatorname{Im} z| \leq \pi$ and $-2\pi/30 \leq \operatorname{Re} z \leq 0$. Then*

$$(1/2)|z| \leq |1 - e^z| \leq |z|.$$

Proof. The extrema of $|1-e^z|/|z|$ lie on the boundary of the rectangular region under consideration; now a consideration of the derivatives of $|1-e^z|/|z|$ on each of the four edges of the region shows that the extrema are at the corners. Computation of the values of $|1-e^z|/|z|$ gives the result. $\square$

Write

$$E^{(3)} = 16\pi^3 m \sum_{n>0} \sum_{\substack{c>0 \\ c=0 \pmod N}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m,n;c)$$

and

$$E_3 = 16\pi^3 m \sum_{n>0} \sum_{\substack{c\leq 2\pi\sqrt{mn} \\ c=0 \pmod N}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m,n;c).$$

So $E^{(2)} = E^{(3)} - E_3$.

The sum E_3, like E_1, is supported in the region where exponential decay dominates. To be precise, the inner sum in E_3 has nonzero terms only when $n \geq (c/2\pi\sqrt{m})^2 \geq N^2/4\pi^2 m$. It follows that

$$\begin{aligned} |E_3| &\leq 16\pi^3 m \sum_{n>N^2/4\pi^2 m} \sum_{\substack{c>0 \\ c=0 \pmod N}} e^{-2\pi n/\sigma N \log N} m^{1/2} c^{-3/2} d(c) \\ &\leq 16\zeta^2(3/2)\pi^3 m^{3/2} (N^{-3/2} d(N)) e^{-N/2\pi m\sigma \log N} (1 - e^{-2\pi/\sigma N \log N})^{-1}. \end{aligned}$$

Using the lower bounds on N and σ, we obtain

Proposition 9. *Suppose $N > 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_3| \leq (8/3)\zeta^2(3/2)\pi^3\sigma m^{3/2}N^{-1/2}\log N d(N)e^{-N/2\pi m\sigma\log N}.$$

It now remains only to bound the main term

$$E^{(3)} = 16\pi^3 m \sum_{n>0} \sum_{\substack{c>0 \\ c=0 \pmod N}} \chi(n)e^{-2\pi n/\sigma N\log N}c^{-2}S(m,n;c)$$

We can write

$$E^{(3)} = 16\pi^3 m \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-2}S(c) \tag{6}$$

where

$$\begin{aligned} S(c) &= \sum_{n>0}\chi(n)e^{-2\pi n/\sigma N\log N}S(m,n;c) \\ &= \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{n>0}\chi(n)e^{-2\pi n/\sigma N\log N}e\left(\frac{mx+ny}{c}\right) \end{aligned}$$

where $e(z) = e^{2\pi iz}$ and $y \in (Z/c\mathbb{Z})^*$ is the multiplicative inverse of x.

For ease of notation, write $A = \sigma N\log N$, and for each integer y write $\epsilon_y = 2\pi(-1/A + yi/c)$. Then

$$\begin{aligned} |S(c)| &\leq \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*} |\sum_{n>0}\chi(n)e^{-2\pi n/A}e\left(\frac{ny}{c}\right)| \\ &= \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*} |\sum_{\alpha=1}^{q}\chi(\alpha)e^{-2\pi\alpha/A}e\left(\frac{\alpha y}{c}\right)\sum_{\nu\geq 0}e^{2\pi q\nu/A}e\left(\frac{q\nu y}{c}\right)| \\ &= \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*} |\sum_{\alpha=1}^{q}\chi(\alpha)e^{-2\pi\alpha/A}e\left(\frac{\alpha y}{c}\right)(1-e^{2\pi q(-1/A+iy/c)})^{-1}| \\ &= \sum_{y\in(\mathbb{Z}/c\mathbb{Z})^*} |(1-e^{q\epsilon_y})^{-1}\sum_{\alpha=1}^{q}\chi(\alpha)e^{\alpha\epsilon_y}| \\ &\leq \sum_{y\in(\mathbb{Z}/c\mathbb{Z})^*} |(1-e^{q\epsilon_y})|^{-1}|\sum_{\alpha=1}^{q}\chi(\alpha)e^{\alpha\epsilon_y}|. \end{aligned}$$

We have the trivial bound $|\sum_{\alpha=1}^{q}\chi(\alpha)e^{\alpha\epsilon_y}| \leq \phi(q)$. (This bound can be sharpened to $O(\sqrt{q}\log q)$ if one wishes to improve the dependence on q.) We now estimate $|\sum_y(1-e^{q\epsilon_y})^{-1}|$. For each y, let $f(y)$ be the unique integer congruent to qy modulo c with $|f(y)| \leq c/2$. By our assumption that $N \not| q$, we have $f(y) \neq 0$. Then by Lemma 8 one has

$$|(1-e^{q\epsilon_y})^{-1}| < \frac{c}{\pi|f(y)|}.$$

Now the values of $|f(y)|$ range over the integers a between 1 and $c/2$ such that $(a,c)=(q,c)$, each of which arises from at most $2(q,c)$ values of y. So we have

$$|\sum_{y\in(\mathbb{Z}/c\mathbb{Z})^*}(1-e^{q\epsilon_y})^{-1}|\leq\frac{2(q,c)c}{\pi}\left[\frac{1}{(q,c)}+\frac{1}{2(q,c)}+\ldots+\frac{1}{r(q,c)}\right]=(2c/\pi)[1+1/2+\ldots+1/r]$$

where r is the largest integer such that $r(q,c)\leq c/2$. The value of $(2c/\pi)[1+\ldots+1/r]$ is largest when $(q,c)=1$; in that case it is bounded above by

$$(2c/\pi)[\log(c/2)+\gamma+2/c].$$

Since $c>400$, the above expression is bounded by $(2/\pi)c\log c$. So, in all, one has

$$|S(c)|<(2/\pi)\phi(q)c\log c. \tag{7}$$

We observe as well that, from the Weil bound, we have

$$|S(c)|\leq\sum_{n>0}e^{-2\pi n/A}m^{1/2}c^{1/2}d(c)\leq m^{1/2}c^{1/2}d(c)(1-e^{-2\pi/A})^{-1}.$$

Recall from the proof of Proposition 6 that $(1-e^{-2\pi/A})^{-1}\leq(1/6)A$ under our conditions on N and σ. So

$$|S(c)|\leq(1/6)Am^{1/2}c^{1/2}d(c). \tag{8}$$

In particular, we immmediately have the following proposition:

Proposition 10. *Suppose $N\geq 400$, $N\not|q$, and $\sigma>q^2/2\pi$. Then*

$$|E^{(3)}|\leq 16\pi^3 m\sum_{\substack{c>0\\c=0\pmod N}}\min[\frac{2}{\pi}\phi(q)c^{-1}\log c,\frac{1}{6}\sigma N\log Nm^{1/2}c^{-3/2}d(c)].$$

This completes the proof of Theorem 1.

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