1 First mid-term exam for Math 204

The exam consists of three questions. You are expected to work on the exam in collaboration with the other members of your group. The task of writing up the problems, however, should be divided among the group, with each member writing up one solution.

The exam is open book, which means not only your book but any other book you care to look at. (However, none of the problems require any mathematics not contained in your textbook.) Calculators, computers, websites, etc. are also fine. The only resource I ask you not to employ is advice from people outside your group.

A successful write-up consists of the following two parts:

1. Your solution to the problem. Since the time pressure is mild, I ask that your solution be written in complete sentences and that you make it clear what theorems you are using (i.e., if you use a theorem from the book, please tell me so and give a reference.)

2. A description of how you came to the solution you wrote down in part 1. Typically, this part will be between 1/2 and 1 page. *It is absolutely acceptable* for the description to be, "Ms. X in my group told me how to do it." If this is the case, you must get Ms. X to explain to you how *she* came by the solution, and you should describe that process in your own words. It is also fine for the description to be, "I found a solution in the book *Introduction to the theory of Y*." If so, your job in part 2 is to describe your personal understanding of the solution so that it is clear to me you did more than copy out an answer.

Please be aware that these problems have many parts, some of which are relatively deep. It's OK not to be able to solve every section of every problem. But *please* give me some indication of what things you tried and what ideas you had.

Grading: Each group will be graded as a whole. We expect to give generous partial credit; we am also open to giving extra credit, if the problems suggest to you further ideas for exploration. We will be less forgiving about sloppiness than we are on ordinary exams–you have plenty of time to check your work.

"How long should I spend on this exam?" We don't have a fixed amount of time in mind. You should think of it as you would writing a paper. You can put as much or little time as you like into it; the more work you put in, the better exam your group will write, and the more your understanding of the material will increase.

The exam is due at the beginning of class on Wednesday, March 8.

This format will be new to most of you. Please don't hesitate to ask us if anything is unclear about what we expect. And now, the questions–good luck!

Problem 1. The building inspector.

You are an engineer who is inspecting a square load-bearing column. The column is subject to certain external temperatures on each of its four faces. Your job is to use the discrete mean-value property, as described in the included handout, to estimate the temperature inside the column. A cross-section of the column is shown below. The temperature at each of the four points A, B, C, and D will be denoted t_1, t_2, t_3, and t_4, while the boundary temperatures on the faces will be denoted b_1, b_2, b_3, and b_4. The discrete-mean value property tells you that at an equilibrium state, we may estimate

$$t_1 = 1/4(b_4 + b_1 + t_2 + t_4). \tag{1}$$

a. Write the equations for t_2, t_3, and t_4 analogous to (1). If $\vec{t}$ denotes the vector

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix},$$

write these equations in the form

$$\vec{t} = M\vec{t} + \vec{b}$$

as in equation (7.2) of the handout. (Note that $\vec{b}$ need not be the vector whose coordinates are b_1, b_2, b_3, and b_4...!)

b. Suppose the boundary temperatures (in degrees Centigrade) are

$$b_1 = 160, b_2 = 80, b_3 = 0, b_4 = 0.$$

What is your estimate for the equilibrium values of the interior temperatures t_1, t_2, t_3, and t_4?

c. The building foreman tells you that the column stands a significant risk of buckling if the temperature difference between any two adjacent points (among the four you studied) is more than about 30 degrees. If you calculated the equilibrium correctly above, you are now extremely nervous. The foreman says he can raise the temperature along face 3 of the column, but that it's very expensive to do so. He asks: by how much would you recommend he increase b_3 in order to make the building safe?

d. The discrete mean-value property technique is an approximation of a physical system by a mathematical model. One way to test the validity of such a model is to test it in situations where our physical experience tells us what the result should be.

From your physical experience, what would you expect the interior temperatures t_1, t_2, t_3, t_4 to be if the boundary temperatures were all 80 degrees? Check that the estimates for t_1, t_2, t_3, and t_4 arrived at by the discrete mean-value property technique coincide with your physical intuition.

If they do not coincide, which do you think is incorrect?

e. From your physical experience, you know that for *any* combination of boundary temperatures (and even if the boundary temperature was not constant along each face!) there must exist an equilibrium state for the interior temperature.

Show that, in accordance with physics, the equation

$$\vec{t} = M\vec{t} + \vec{b}$$

has a solution for *any* $\vec{b} \in \mathbb{R}^4$.

f. (optional) Make up some other shape and fix some temperature distribution on the boundary of the shape, and choose some points in the interior where you would like to estimate the temperature. Make a note of your mental estimates; then estimate the equilibrium temperature at various points in the interior via the discrete mean-value property technique. In other words, write (and answer) your own version of part **b** of this exam problem. Please use a system containing at least three interior points. If you are handy with a computer algebra system like Maple or Mathematica, it would be great to analyze a system with ten or more points.

How did your initial estimates compare with those you obtained by linear algebra? What happens if the temperature is constant on the boundary? (Compare with **d** above.) Does $\vec{t} = M\vec{t} + \vec{b}$ still have a solution for any $\vec{b}$?

Problem 2. Cubic interpolation

Cubic interpolation is a method for producing a smooth curve which bends in a given direction, or which passes through a given set of points. This procedure, and its more sophisticated cousins the cubic spline and the Bezier technique, are used by engineers (to model things like bending steel rods), by designers (to make nice-looking curves for things like car windshields and typefaces), and by movie-makers. (For instance, a computer-generated alien might initially be represented as a stack of polyhedra; to "smooth the corners" and make the alien look more fleshy, one needs an algorithm for producing smooth curves and surfaces approximating a given set of data.)

If you're interested in learning more about splining, you might look at the website `http://moshplant.com/direct-or/bezier/index.html`. For a linear algebra challenge (not part of this exam!) you could try to derive on your own the formulas that appear there.

a. Draw the quadrilateral bounded by the points $(0,0),(1,4),(2,3)$, and $(3,0)$. This might be an alien's humped back, as represented by a finite set of points. Smooth the alien by finding a cubic polynomial f such that $f(0)=0, f(1)=4, f(2)=3$, and $f(3)=0$. Sketch this curve; or, if you have a computer graphics program at hand, print out a graph of the curve.

b. Define a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ by the following rule. Suppose

$$\vec{v} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Then

$$T(\vec{v}) = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix},$$

where

$$f = c_0 + c_1 x + c_2 x^2 + c_3 x^3.$$

So, for example,

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 9 \\ 28 \end{bmatrix}.$$

Find the 4×4 matrix A such that

$$T(\vec{v}) = A\vec{v}.$$

c. What is the rank of A? Prove that, for any real numbers a, b, c, and d, there exists a cubic polynomial f whose graph passes through the points $(0, a), (1, b), (2, c)$, and $(3, d)$, and that this polynomial is unique.

d. Suppose we only had three points, $(0, a), (1, b)$, and $(3, c)$ to work with. Carry out an analysis similar to the above to answer the questions: is there always a cubic polynomial whose graph passes through the three points? Is this cubic polynomial unique?

e. Let V be the subspace of P^3 consisting of those cubic polynomials f such that $f(0) = 0, f(1) = 0$, and $f(2) = 0$. Give a basis for V. Describe the set of cubic polynomials f such that $f(0) = 0, f(1) = 1$, and $f(2) = 2$.

Problem 3. Ranks and products

Remember that an $n \times n$ matrix is invertible if and only if its rank is n. (Strang, p.96) We know that if two matrices A and B are invertible, then their product is also invertible; in fact, the inverse of AB is just $B^{-1}A^{-1}$. We can rephrase this fact as follows: if $\text{rank}A = \text{rank}B = n$, then the rank of AB is also n. In this problem, we will investigate the problem of computing the rank of AB in case the ranks of A and B are smaller than n.

a. Find examples, or prove that no examples exist, of 3×3 matrices A and B such that $\text{rank}(A) = \text{rank}(B) = 2$ and

- $\text{rank}AB = 3$;
- $\text{rank}AB = 2$;
- $\text{rank}AB = 1$;
- $\text{rank}AB = 0$.

From now on, we do not assume that A and B are 3×3 matrices, or even necessarily square matrices. (However, in order to get ideas for the proofs, you may find it useful to experiment with 3×3 or 3×2 or 2×2 or even 1×2 matrices!)

b. On pp.5-6 of the posted lecture notes for Week 3 on the CourseInfo web site

`http://courseinfo.Princeton.EDU/courses/MAT204_S2000/`

there is a proof that, if A is an *invertible* $n \times n$ matrix and B is any $n \times p$ matrix, then

$$\text{rank}(AB) = \text{rank}(B).$$

Prove the more general fact that, for any $m \times n$ matrix A and any $n \times p$ matrix B,

$$\text{rank}(AB) \leq \text{rank}(B).$$

c. Part **b** gives you an upper bound for the rank of AB; we might also want to find a lower bound. By experimenting with different matrices of different sizes, develop a conjecture: If A is an $m \times n$ matrix and B is an $n \times p$ matrix, what is the smallest possible value of $\text{rank}(AB)$ in terms of the $\text{rank}(A)$ and $\text{rank}(B)$?

Prove your conjecture.

If you find yourself stuck, it might be instructive to consider the following more limited problem: for which values of $\text{rank}(A)$ and $\text{rank}(B)$ is it possible that $\text{rank}(AB) = 0$?

Problem 4: Elk manager

In this problem, you are managing a herd of elk in the Great Plains. The elk are distributed between Yellowstone and Grand Teton (both in Wyoming), and the Black Hills (in South Dakota). Each year, 40% of the Yellowstone elk migrate to nearby Grand Teton, and 30% of the Grand Teton elk to Yellowstone. Meanwhile, 5% of the Yellowstone and 5% of the Grand Teton elk migrate all the way to the Black Hills; of the Black Hills elk, 5% migrate to Grand Teton and 5% to Yellowstone.

For example; suppose that at the beginning of the year there are 600 elk in Yellowstone, 500 in Grand Teton, and 300 in the Black Hills. By the end of the year, 240 of the Yellowstone elk will have decamped for Grand Teton, and 30 for the Black Hills. On the other hand, Yellowstone accepts 150 elk from Grand Teton and 15 from the Black Hills. In sum, Yellowstone ends the year with 495 elk. Similarly, one can calculate that the year ends with 580 elk in Grand Teton and 325 in the Black Hills.

a We can represent the population distribution of the elk herd at any given time by a vector $\vec{v}_t$ in $\mathbb{R}^3$, where

$$\vec{v}_t = \begin{bmatrix} \text{number of elk at Yellowstone in year } t \\ \text{number of elk at Grand Teton in year } t \\ \text{number of elk at the Black Hills in year } t \end{bmatrix}.$$

Find a 3×3 matrix A such that

$$\vec{v}_{t+1} = A\vec{v}_t.$$

For example, your matrix A should satisfy

$$A \begin{bmatrix} 600 \\ 500 \\ 300 \end{bmatrix} = \begin{bmatrix} 495 \\ 580 \\ 325 \end{bmatrix}.$$

b A population distribution $\vec{v}$ is *stable* if it does not change from one year to the next; that is,

$$A\vec{v} = \vec{v},$$

or, equivalently,

$$(I - A)\vec{v} = \vec{0}.$$

Find such a $\vec{v}$, other than the zero vector ("the extinction equilibrium"). Describe the set of all stable $\vec{v}$. What is the rank of $I - A$?

c At present, the elk have settled into a stable state, which, as you have seen above, means that the number of elk in each park are roughly equal. Park Ranger Shapiro informs you that, because of increased human presence in Wyoming, he would like to see at least half the elk living in the Black Hills. Shapiro proposes to artificially increase the migration rate from Grand Teton to the Black Hills by busing elk from one park to the other. That is, the migration rate will be increased from 5% per year to $(5+u)\%$, for some u between 0 and 95. This will change A to some new matrix A'. The new stable state will be a vector $\vec{v}$ such that $A'\vec{v} = \vec{v}$.

How large does u need to be to achieve Shapiro's goal?

d Suppose that the migration rates in the introduction were replaced by some other set of migration rates, and let A be the matrix describing the resulting system.

Show that the $I - A$ is not invertible, no matter what the migration rates are. One way to break the problem up would be as follows. Step 1: show that

$$(I - A)\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

does not have a solution. Step 2: show that it follows from the result of Step 1 that $I - A$ is not invertible.