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Bijective Quasi-Isometries of Amenable Groups

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Abstract. Whyte showed that any quasi-isometry between non-amenable groups is a bounded distance from a bijection. In contrast this paper shows that for amenable groups, inclusion of a proper subgroup of finite index is never a bounded distance from a bijection.

1. Introduction

In his book on infinite groups, Gromov [**Gr**, page 23] asked whether inclusion of finite index subgroups $F_m \longrightarrow F_n$ of two free groups is a bounded distance from a bi-Lipschitz map (i.e. a bijective quasi-isometry). Papasoglu answered this question affirmatively in [**Pa**]. A more general question asked in [**H**, page 107] is whether any two groups which are quasi-isometric always have a bijective quasi-isometry between them? No counterexamples have been found.

Whyte showed in [**Wh**] that any quasi-isometry between *non-amenable* groups is a bounded distance from a bijection. In contrast, we will show that for *amenable* groups, inclusion of a finite index proper subgroup is never a bounded distance from a bijection (see Theorem 3.5). However, if the subgroup admits an "n-to-1" self quasi-isometry (where n is the index of the subgroup) one can always compose this self map with the subgroup inclusion map to get a quasi-isometry that is a bounded distance from a bijective quasi-isometry. So for such groups commensurability implies that there does exist a bijective quasi-isometry.

2. Preliminaries

Definition 2.1. A map between metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there exist $C, K \geq 0$ such that for all $x, y \in X$

$$-C + \frac{1}{K} d(x,y) \leq d(f(x), f(y)) \leq K d(x,y) + C \tag{2.1}$$

and there also exist $g : Y \longrightarrow X$ satisfying 2.1 such that $f \circ g$ and $g \circ f$ are a bounded distance from the identity. In this case X and Y are said to be *quasi-isometric.*

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Definition 2.2. A metric space X is a *uniformly discrete space of bounded geometry* (UDBG space) if

- there is an $\epsilon > 0$ such that, for all $x, y \in X$, $d(x,y) < \epsilon \Rightarrow x = y$ (i.e. X is uniformly discrete), and
- for any $r > 0$ there is a bound M_r on the size of any r-ball in X (i.e. X has bounded geometry).

Example 2.3. A finitely-generated group with the word metric $d(x,y) = ||x^{-1}y||$ is a main example of a UDGB-space. The norm is defined with respect to a specific generating set so two different generating sets may give metric spaces which are not isometric but they are quasi-isometric.

Example 2.4. Any proper metric space admitting a properly discontinuous isometric cocompact group action is also a UDBG-space.

To define the notion of an *amenable* space we need some notation. Let $|S|$ denote the cardinality of the set S and $\partial_r S = \{x \in X \mid 0 < d(x,S) \leq r\}$ denote the r-boundary of S.

Definition 2.5 (Følner Criterion). A UDBG-space is *amenable* if there exists a sequence of finite subsets $\{S_i\}$ such that for all $r > 0$

$$\lim_{i \to \infty} \frac{|\partial_r S_i|}{|S_i|} = 0$$

Such a sequence is called a *Følner sequence.*

We say a finitely generated group is amenable if it is amenable as a UDBG-space with the word metric. Since amenability is preserved by quasi-isometries this is a well defined notion.

It will also be useful to review some conventions of big "O" notation.

Definition 2.6. For two real valued functions f, g we say

- $f(i) = O(g(i))$ if there exist $C, K > 0$, such that $|f(i)| \leq C|g(i)|$ for all $i \geq K$.
- $O(f(i)) < O(g(i))$ if $g(i) \neq O(f(i))$
- $O(g(i)) = O(f(i))$ if both $g(i) = O(f(i))$ and $f(i) = O(g(i))$.

One important property we will use is that for $f, g \geq 0$

$$O(f(i) + g(i)) = O(f(i)) \text{ if } g(i) = O(f(i)).$$

Using this notation, the statement of amenability can be rephrased as follows:

X is amenable if there exists a sequence of finite sets $S_i \subset X$ such that $O(|\partial_r S_i|) < O(|S_i|)$ for all $r > 0$.

3. Uniformly finite homology

The uniformly finite homology groups $H_i^{uf}(X)$ were first introduced by Block-Weinberger in [**BW1**]. Only $H_0^{uf}(X;\mathbb{Z})$ is needed here, but for a detailed discussion of uniformly finite homology, see [**Wh**] or [**BW1**].

For a UDBG-space X, let $C_0^{uf}(X)$ denote the vector space of infinite formal sums of the form

$$c = \sum_{x \in X} a_x x \quad (a_x \in \mathbb{Z})$$

where there exists $M_c > 0$ such that $|a_x| \leq M_c$. Let $C_1^{uf}(X)$ denote the vector space of infinite formal sums of the form

$$c = \sum_{x,y \in X} a_{(x,y)}(x,y)$$

where there exist $M_c > 0$ such that $|a_{(x,y)}| < M_c$ and $R_c > 0$ such that $a_{(x,y)} = 0$ if $d(x,y) > R_c$. Define a boundary map by

$$\begin{array}{rcl} \partial : C_1^{uf}(X) & \longrightarrow & C_0^{uf}(X) \\ (x,y) & \longmapsto & y - x \end{array}$$

and extending by linearity. Then we let

$$H_0^{uf}(X) = C_0^{uf}(X)/\partial(C_1^{uf}(X))$$

Some important facts about $H_0^{uf}(X)$ we will not prove here (see [**Wh**]) are

- if X and Y are quasi-isometric then $H_0^{uf}(X) \cong H_0^{uf}(Y)$.
- if X is infinite then $H_0^{uf}(X)$ is a vector space over $\mathbb{R}$. (When X is finite $H_0^{uf}(X) \cong \mathbb{Z}$)

Definition 3.1. Any subset $S \subset X$ defines a class in $H_0^{uf}(X)$, denoted $[S]$, where [S] is the class of the chain $\sum_{x \in S} x$. We call $[X]$ the *fundamental class* of X in $H_0^{uf}(X)$.

Using uniformly finite homology, Whyte developed in [**Wh**] a test to determine when a quasi-isometry between UDBG spaces is a bounded distance from a bijection.

Theorem 3.2 (Whyte). [**Wh**] *Let $f : X \to Y$ be a quasi-isometry between UDBG-spaces. Then there exists a bijective map a bounded distance from f if and only if $f_*([X]) = [Y]$. (Here $f_*([X]) = [\sum_{x \in X} f(x)]$)*

For non-amenable spaces we have the following theorem:

Theorem 3.3 (Block-Weinberger). [**BW1**] *Let X be a UDBG-space. Then the following are equivalent:*

- *X is non-amenable.*
- *$H_0^{uf}(X) = 0$*
- *there exists $c = \sum_{x \in X} a_x x \in C_0^{uf}(X)$ with $a_x > 0$ such that $[c] = 0$ in $H_0^{uf}(X)$*

Some of the motivation behind Whyte's Theorem 3.2 was that combined with Theorem 3.3 it implies:

Any quasi-isometry between finitely generated non-amenable groups is a bounded distance from a bijection.

We can also use Theorem 3.2 to investigate quasi-isometries of amenable groups. To use Theorem 3.2 we need to be able to check when a chain in $c \in C_0^{uf}(X)$ represents the zero class in $H_0^{uf}(X)$. The following theorem gives such a criterion.

THEOREM 3.4 (Block-Weinberger). [**BW1**] *(Theorem 7.6 in* [**Wh**]*) Let X be a UDBG-space, and let $c = \sum_{x\in X} a_x x \in C_0^{uf}(X)$. Then we have $[c] = 0 \in H_0^{uf}(X)$ if and only if there exist an r such that for any Følner sequence $\{S_i\}$,*

$$\left|\sum_{x\in S_i} a_x\right| = O(|\partial_r S_i|).$$

We now show how Whyte's criterion can be used to show that subgroup inclusion for amenable groups is not a bounded distance from a bijection.

THEOREM 3.5. *Let G be an amenable group with proper subgroup H of finite index, i.e. $[G : H] = n > 1$. Then the inclusion map $i : H \hookrightarrow G$ is not a bounded distance from a bijective map.*

PROOF. Using Theorem 3.4 we show that the chain $c = \sum_{x\in G\setminus H} x$ gives a nonzero class in $H_0^{uf}(G)$, that is $[c] = [G] - [H] \neq 0$. To this end let $\{S_i\}$ be any Følner sequence for G. Now $G = \bigcup_{k=1}^{n} g_k H$ and

$$\sum_{k=1}^{n} |S_i \setminus g_k H| = (n-1)|S_i|$$

so

$$O(|S_i|) = O((n-1)|S_i|) = O(\sum_{k=1}^{n} |S_i \setminus g_k H|) = O(|S_i \setminus g_{k_i} H|)$$

for some k_i. Let $F_i = g_{k_i}^{-1} S_i$. These sets also form a Følner sequence, since left multiplication by $g_{k_i}^{-1}$ is an isometry. Now $|g_{k_i}^{-1} S_i \setminus H| = |S_i \setminus g_{k_i} H|$. This gives us

$$O(F_i \setminus H) = O(|F_i|) > O(|\partial_r F_i|).$$

for any $r > 0$. So for the chain c,

$$\left|\sum_{x\in F_i} a_x\right| = |F_i \setminus H| = O(|F_i|) \neq O(|\partial_r F_i|),$$

and so $[c] \neq 0$. □

The following is a shorter proof of Theorem 3.5 suggested by Weinberger.

PROOF. First note that $[G] = n[H]$. To see this express G as a disjoint union of n right cosets of H. Now since right multiplication is a bounded distance from the identity map we have $[H] = [Hg_i]$. This gives

$$[G] = \left[\bigsqcup_{i=1}^{n} Hg_i\right] = \sum_{i=1}^{n}[Hg_i] = n[H].$$

Now for the inclusion map i to be a bounded distance from a bijection, Theorem 3.2 tells us that we need

$$[H] = i_*([H]) = [G].$$

So what we really need is $[H] = n[H]$. Now G is amenable so $[H] \neq 0$. And since $H_0^{uf}(G)$ is torsion free $[H] = n[H]$ only if $n = 1$. □

Theorem 3.5 is actually a corollary of a more general result.

THEOREM 3.6. *If $\phi : H \longrightarrow G$ is a homomorphism of amenable groups with finite index image, $[G : \phi(H)] = n$, and finite kernel, $|\phi^{-1}(0)| = k$, then ϕ is a bounded distance from a bijection if and only if $n = k$.*

PROOF. As above, to get a bijection we need $\phi_*([H]) = [G]$. But $\phi_*([H]) = k[\phi(H)]$ and $[G] = n[\phi(H)]$. Now G is an amenable group, so $[\phi(G)] \neq 0$, giving us $n = k$. □

4. "n-to-1" self Quasi-Isometries

In this section we show how one can use subgroup inclusion to get bijective quasi-isometries between certain groups.

DEFINITION 4.1. We call $f : X \longrightarrow X$ an *"n-to-1" self quasi-isometry* if f is a quasi-isometry and $|f^{-1}(x)| = n$ for all $x \in X$. In this case $f_*([X]) = n[X]$.

Theorem 3.6 suggests that we may be able to "fix" subgroup inclusion $i : H \to G$ of index n by precomposing with an n-to-1 quasi-isometry $f : G \to G$. Then $i \circ f : H \longrightarrow G$ is a new quasi-isometry that is a bounded distance from a bijection, since

$$(i \circ f)_*([H]) = i_*(n[H]) = n[H] = [G].$$

This leads to the question:

Which amenable groups admit "n-to-1" quasi-isometries?

EXAMPLE 4.2. It is easy to define an "n-to-1" quasi-isometry of $\mathbb{Z}$

$$\begin{aligned} \phi : \quad & \mathbb{Z} \to \mathbb{Z} \\ & k \mapsto \lfloor \tfrac{k}{n} \rfloor \end{aligned}$$

where $\lfloor \frac{k}{n} \rfloor$ denotes the greatest integer less than or equal to $\frac{k}{n}$. This is an "n-to-1" map of $\mathbb{Z}$ which is a $(\frac{1}{n}, 1)$ quasi-isometry. One can extend this idea to get an "n-to-1" map on $\mathbb{Z}^m$ by applying the above map to one of the coordinates.

$$\phi(k_1, k_2, \cdots, k_m) = (\lfloor \frac{k_1}{n} \rfloor, k_2, \cdots, k_m)$$

We now consider another class of examples.

EXAMPLE 4.3. The *solvable Baumslag Solitar groups* are given by the presentation

$$BS(1, m) = \langle a, b \mid aba^{-1} = b^m \rangle .$$

We can view $BS(1, m)$ as a union of cosets of the subgroup $\langle b \rangle \cong \mathbb{Z}$. By identifying each coset with $\mathbb{Z}$ we can define an "n-to-1" map in a similar way as we do for $\mathbb{Z}$

$$f_\alpha : \alpha b^i \mapsto \alpha b^{\lfloor \frac{i}{n} \rfloor}$$

where α is the coset representative. Picking a set C of coset representatives gives us an "n-to-1" map of $BS(1, m)$

$$f_C : g \mapsto b^{\lfloor \frac{i}{n} \rfloor} \alpha$$

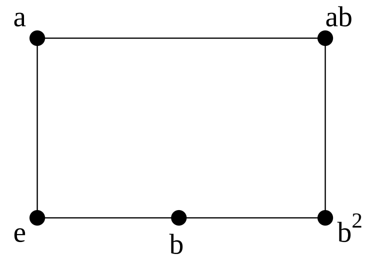

FIGURE 1. the horobrick h_2 which is the fundamental domain corresponding to the element a in $BS(1,2)$

where $g = \alpha b^i$ for some $\alpha \in C$.

A priori, if we choose random coset representatives for each coset we may not get a quasi-isometry of $BS(1,m)$. It turns out that we can identify the cosets with $\mathbb{Z}$ in such a way so that the resulting map is actually a quasi-isometry.

In order to understand how to pick the coset representatives it is useful to review some ideas from [**FM**]. The group $BS(1,m)$ acts properly discontinuously and cocompactly on a metric 2-complex X_m, which is a warped product of a tree T_m and $\mathbb{R}$. The tree T_m is a $(m+1)$-valent directed tree with one "incoming edge" and m "outgoing edges" at each node. There is a natural projection $X_m \to T_m$. The inverse image of a coherently oriented line, (a bi-infinite path in T_m respecting the orientation) is a hyperbolic plane. Any time we refer to these embedded hyperbolic planes we will identify them with the upper half plane model of H^2. The inverse image of a vertex is a horocycle, (called a branching horocycle). We can pick the basepoint of X_m to lie on a branching horocycle.

X_m is actually the universal cover of a complex C_m (see [**FM**] for details) whose fundamental group is $BS(1,m)$. The fundamental domain can be thought of as a "horobrick" $h_m \subset H^2$ defined by the region bounded by $0 \leq x \leq n$ and $1 \leq y \leq m$ so that the top of the horobrick has length 1 and the bottom has length m (see figure 1).

We can define a quasi-isometry $i : BS(1,m) \to X_m$ by mapping e, the identiy of $BS(1,m)$, to the basepoint of X_m and extending equivariantly. Then we can view $BS(1,m)$ as embedded in X_m where group elements lie on branching horocycles and each branching horocycle contains a coset of $\langle b \rangle$. Elements which differ only by the generator a are distance $log(m)$ apart and lie on adjacent branching horocycles. Since i is a quasi-isometry it has a coarse inverse. Let $j : X_m \to BS(1,m)$ be the coarse inverse of i which maps each fundamental domain in X_m to the unique element of $BS(1,m)$ in that domain. (i.e. each horobrick is mapped to the element in the upper left corner.) Any map $f : BS(1,m) \to BS(1,m)$ gives us a map $i \circ f \circ j : X_m \to X_m$ which is a quasi-isometry of X_m if and only if f is a quasi-isometry of $BS(1,m)$. When convenient we will make no distinctions between f and $i \circ f \circ j$.

The key idea is to "line up" all of the cosets so that our map f, when restricted to each hyperbolic plane in X_m, is a bounded distance from the quasi-isometry

$$\phi : (x,y) \longmapsto (\frac{1}{n}x, y).$$

To this end we need to consider another projection $X_m \to H^2$. From [**FM**] we know that there exists a unique map $\rho_m : X_m \to H^2$ with the following properties:

- ρ_m takes horocycles to horocycles
- ρ_m is an isometry when restricted to each hyperbolic plane in X_m
- ρ_m is normalized to take the base point of X_m to the point $(x, y) = (0, 1)$

Let $l = \{(0, y) \in H^2\}$ be the y-axis and consider $T = \rho^{-1}(l)$. T intersects each branching horocycle at exactly one point. This will be our reference point which we will call "0". Since each branching horocycle "contains" a coset of $\langle b \rangle$, pick the coset representative for this coset to be a group element α which lies closest to "0". (There may be two such elements in some cases). Note that for each α the distance between α and T is at most one. Our map $i \circ f_\alpha \circ j$ is bounded distance from the map ϕ when restricted to each branching horocycle. Because all of the α's are a uniformly bounded distance from T we have that for each hyperbolic plane Q in X_m the total map f_C restricted to Q is a bounded distance from ϕ. So by the rubberband principle $i \circ f_C \circ j$ is a quasi-isometry of X_m and so f_C is an "n-to-1" quasi-isometry of $BS(1, m)$.

We now consider one criterion for when a group does admit an "n-to-1" quasi-isometry. If G contains a subset G' such that

$$G = \bigsqcup_{i=1}^{n} G' g_i.$$

and there exists a bijective quasi-isometry

$$f : G' \longrightarrow G$$

then f extends to an "n-to-1" self quasi-isometry of G given by

$$\begin{array}{rcl} f' : G & \longrightarrow & G \\ g' g_i & \longmapsto & f(g') \quad (g' \in G') \end{array}$$

This holds in particular if G' is a subgroup of G with $|G : G'| = n$ and $f : G' \to G$ is an isomorphism.

If it were possible to find an "n-to-1" self quasi-isometry for all amenable groups, then we would have a bijective quasi-isometry between any two commensurable, amenable groups.

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References

[BW1] Block, J.,Weinberger, S., *Aperiodic tilings, positive scalar curvature and amenability of spaces* J. Amer. Math. Soc. 5 (1992), no. 4, p. 907-918

[BW2] Block, J., Weinberger, S., *Large scale homology theories and geometry* AMPS/IP Studies in Advanced Mathematics, 2 1997. p. 522-569

[FM] Farb, B., Mosher, L., *A rigidity theorem for the solvable Baumslag-Solitar groups*, Inventiones mathematicae, Springer-Verlag 1998. p. 419-451

[Gr] Gromov, M. *Asymptotic invariants of infinite groups*, in Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, Cambridge Univ. Press, Cambridge, 1993

[H] P. de la Harpe, *Topics in Geometric Group Theory* Chicago, 2000

[Pa] Papasoglu, P., *Homogeneous trees are bi-Lipschitz equivalent* Geom. Dedicata 54 (1995), no. 3, p. 301-306
[Wh] Whyte, K., *Amenability, Bilipschitz Equivalence, and the Von Neumann Conjecture* Duke Journal of Mathematics 1999, p. 93-112

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