EXERCISES

Let R be a ring with 1.

1. Let $f : R \to S$ be a ring homomorphism from the ring R to the ring S. Verify the details that $sr = sf(r)$ defines a right R-action on S under which S is an (S, R)-bimodule.
2. Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.
3. Show that $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ and $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.
4. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]
5. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.
6. If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.
7. If R is any integral domain with quotient field Q and N is a left R-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in Q$ and some $n \in N$.
8. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $U = R^\times$ be the set of nonzero elements in R and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements (u, n) with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n)$ if and only if $u'n = un'$ in N.
   (a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Section 4 of Chapter 15, cf. also Section 5 in Chapter 7.]
   (b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R$, $b \in U$, $n \in N$, is an R-balanced map, so induces a homomorphism f from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u, n)}) = (1/u)\otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.
   (c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.
   (d) If A is an abelian group, show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ if and only if A is a torsion abelian group (i.e., every element of A has finite order).
9. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $Q \otimes_R N$ be the module obtained from N by extension of scalars from R to Q. Prove that the kernel of the R-module homomorphism $\iota : N \to Q \otimes_R N$ is the torsion submodule of N (cf. Exercise 8 in Section 1). [Use the previous exercise.]
10. Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.
    (a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.
    (b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}$, $n = 1$, $M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]
11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

12. Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.
13. Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)$ be $a_1b_1 + \cdots + a_nb_n$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.
14. Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-module. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\oplus_{i\in I} N_i) \cong \oplus_{i\in I}(M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 20, Section 3. [Use the same argument as for the direct sum of two modules, taking care to note where the direct *sum* hypothesis is needed — cf. the next exercise.]
15. Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \ldots$.]
16. Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.
    (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.
    (b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.
17. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R-module annihilated by both 2 and x.
    (a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by
    $$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \frac{a_0}{2}b_1 \bmod 2$$
    is R-bilinear.
    (b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2}q'(0)$ where q' denotes the usual polynomial derivative of q.
    (c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.
18. Suppose I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements (i.e., $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies that $m = 0$).
19. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$ as in Exercise 17. Show that the nonzero element $2 \otimes x - x \otimes 2$ in $I \otimes_R I$ is a torsion element. Show in fact that $2 \otimes x - x \otimes 2$ is annihilated by both 2 and x and that the submodule of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$ is isomorphic to R/I.
20. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.
21. Suppose R is commutative and let I and J be ideals of R.
    (a) Show there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to the element ij.
    (b) Give an example to show that the map in (a) need not be injective (cf. Exercise 17).
22. Suppose that M is a left and a right R-module such that $rm = mr$ for all $r \in R$ and $m \in M$. Show that the elements r_1r_2 and r_2r_1 act the same on M for every $r_1, r_2 \in R$. (This explains why the assumption that R is commutative in the definition of an R-algebra is a fairly natural one.)
23. Verify the details that the multiplication in Proposition 19 makes $A \otimes_R B$ into an R-algebra.