Math 542 Homework 9 - Due Thursday April 18

(1) Let $z = e^{2\pi i/p} = \cos(2\pi/p) + i\sin(2\pi/p) \in \mathbb{C}$ where p is a prime. Find $[\mathbb{Q}(z) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(z)$ over $\mathbb{Q}$.

(2) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find $[K : \mathbb{Q}]$ and find a basis for K over $\mathbb{Q}$.

(3) Suppose that E/F is an extension and $char(F) \neq 2$. Show that
$$[E : F] = 2 \Longleftrightarrow \exists u \in E \text{ so that } E = F(u), u \notin F, u^2 \in F$$

(4) Suppose that $u \in \mathbb{R}$. Then u is said to be *constructible* if there exists a sequence $F_0, F_1, \ldots, F_k$ of subfields of $\mathbb{R}$ so that $F_0 = \mathbb{Q}$, $u \in F_k$
$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$$
and $[F_i : F_{i-1}] = 2$ for $i = 1, \ldots, k$.

(a) Show that $\sqrt{2 + \sqrt{1 + \sqrt{5}}}$ is constructible.

(b) Show that $\cos(\pi/9)$ is not constructible. [1]

(5) (a) Find all irreducible polynomials of degree ≤ 4 over $\mathbb{F}_2$.

(b) Give examples of fields of order 2,4,8 and 16.

[1] This solves the ancient problem of whether it is possible to trisect the angle $\pi/3$ using a ruler and a compass. If this were possible then one could show that $\cos(\pi/9)$ is constructible. (See your book for more info).