# Math 542 Homework 8 - Due Thursday April 11

(1) Suppose that $E/F$ is an extension so that $[E:F]=1$. Show that $E=F$.

(2) Let $u=\cos(\pi/9)\in\mathbb{C}$. Show that $u$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial of $u$ over $\mathbb{Q}$.

(3) (a) If $F$ is a field and $p=char(F)$ (see notes) show that either $p$ is 0 or $p$ is a prime.
    (b) If $E$ is an extension of $\mathbb{Q}$ what is $char(E)$?
    (c) Let $p$ be a prime and let $\mathbb{F}_p=\mathbb{Z}/(p)=\{\bar{0},\bar{1},\dots,\overline{p-1}\}$ be the field of integers mod $p$. What is $char(\mathbb{F}_p)$?

(4) Let $r=\sqrt[3]{5}$. Find the inverse of $1+r-r^2$ in $\mathbb{Q}(r)$. Express your answer in the form $a_0+a_1r+a_2r^2$ where $a_0,a_1,a_2\in\mathbb{Q}$.

(5) Suppose that $E/F$ is an extension. Let $K$ be the set of all elements of $E$ which are algebraic over $F$. Show that $K$ is a subfield of $E$. (In the case when $E/F$ is $\mathbb{C}/\mathbb{Q}$, $K$ is called the field of algebraic numbers.)

# Math 542 Homework 9 - Due Thursday April 18

(1) Let $z = e^{2\pi i/p} = \cos(2\pi/p) + i\sin(2\pi/p) \in \mathbb{C}$ where $p$ is a prime. Find $[\mathbb{Q}(z):\mathbb{Q}]$ and find a basis for $\mathbb{Q}(z)$ over $\mathbb{Q}$.

(2) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find $[K:\mathbb{Q}]$ and find a basis for $K$ over $\mathbb{Q}$.

(3) Suppose that $E/F$ is an extension and $char(F) \neq 2$. Show that

$$[E:F] = 2 \iff \exists u \in E \text{ so that } E = F(u), u \notin F, u^2 \in F$$

(4) Suppose that $u \in \mathbb{R}$. Then $u$ is said to be *constructible* if there exists a sequence $F_0, F_1, \ldots, F_k$ of subfields of $\mathbb{R}$ so that $F_0 = \mathbb{Q}$, $u \in F_k$

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$$

and $[F_i : F_{i-1}] = 2$ for $i = 1, \ldots, k$.

(a) Show that $\sqrt{2 + \sqrt{1 + \sqrt{5}}}$ is constructible.

(b) Show that $\cos(\pi/9)$ is not constructible. [1]

(5) (a) Find all irreducible polynomials of degree $\leq 4$ over $\mathbb{F}_2$.

(b) Give examples of fields of order 2,4,8 and 16.

[1]This solves the ancient problem of whether it is possible to trisect the angle $\pi/3$ using a ruler and a compass. If this were possible then one could show that $\cos(\pi/9)$ is constructible. (See your book for more info).

# Math 542 Homework 10 - Due Thursday April 25

(1) Let $f(\lambda) \in F[\lambda]$ be a monic polynomial of degree $n \geq 1$.
Let $E$ be the splitting field of $f(\lambda)$ over $F$ and suppose that $f(\lambda)$ has distinct roots in $E$. Let $K_1$ and $K_2$ be subfields of $E/F$ so that[2] $\langle K_1, K_2 \rangle = E$ and

$$[K_1 : F][K_2 : F] = [E : F].$$

Let

$$G = Gal(E/F), \; G_1 = Gal(E/K_1) \text{ and } G_2 = Gal(E/K_2).$$

(So $G_1$ and $G_2$ are subgroups of $G$.)

Show that

a) $G_1 \cap G_2 = \{\epsilon\}$

b) $G_1 G_2 = G$.[3]

(2) Suppose that $E/F$ is a finite extension of degree $n$.

a) Let $E'/F'$ be an extension and let $\phi : F \to F'$ be an isomorphism. Show that the number of extensions of $\phi$ to a homomorphism of $E$ to $E'$ is $\leq n$.

b) Show that $|Gal(E/F)| \leq n$.

(3) Let $f(\lambda) \in \mathbb{Q}[\lambda]$ be given below. Let $E$ be the splitting field of $f(\lambda)$ over $\mathbb{Q}$ and let $G = Gal(E/\mathbb{Q})$. Find $E$ and find $[E : \mathbb{Q}]$.

a) $f(\lambda) = \lambda^5 - 2 \in \mathbb{Q}[\lambda]$

b) $f(\lambda) = \lambda^4 - 4\lambda^2 + 2 \in \mathbb{Q}[\lambda]$ (Hint: Show that $E$ is generated by a single root of $f(\lambda)$ in $E$.)

(4) Let $E = \mathbb{F}_2(\lambda)/(q(\lambda))$ where $q(\lambda) = \lambda^4 + \lambda + 1 \in \mathbb{F}_2(\lambda)$. Then

$$E = \{a_0 + a_1 r + a_2 r^2 + a_3 r^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{F}_2\}$$

where $r = \lambda + (q(\lambda))$. From our previous assignment we know that $E$ is a field of order 16. Find a generator for the group $E^\times = \{u \in E \mid u \neq 0\}$.

[2] The field generated by $K_1$ and $K_2$ denoted $\langle K_1, K_2 \rangle$ is the smallest subfield of $E$ containing $K_1, K_2$.

[3] Recall $G_1 G_2 = \{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\}$. In general it is not a subgroup.

# Math 542 Homework 11 - Due Thursday May 2

(1) Let $f(\lambda) = \lambda^4 - 4\lambda^2 + 2 \in \mathbb{Q}[\lambda]$, let $E$ be the splitting field of $f(\lambda)$ over $\mathbb{Q}$ and let $G = Gal(E/\mathbb{Q})$.

  a) Find $E$ and $[E : \mathbb{Q}]$.
  b) Find $G$ as a group of permutations of the roots of $f(\lambda)$.
  c) Find all of the subgroups of $G$. Which of there subgroups are normal in $G$?
  d) For each subgroup of $G$, find the corresponding subfield of $E/\mathbb{Q}$, given generators for the subfield and indicate whether or not the subfield is a normal extension of $\mathbb{Q}$.

(2) Let $f(\lambda) = \lambda^3 - 5$, let $E$ be the splitting field of $f(\lambda)$ over $\mathbb{Q}$ and let $G = Gal(E/\mathbb{Q})$.

  a) Find $E$ and $[E : \mathbb{Q}]$.
  b) Find $G$ as a group of permutations of the roots of $f(\lambda)$.
  c) Find all of the subgroups of $G$. Which of there subgroups are normal in $G$?
  d) For each subgroup of $G$, find the corresponding subfield of $E/\mathbb{Q}$, given generators for the subfield and indicate whether or not the subfield is a normal extension of $\mathbb{Q}$.