MATH 542 HOMEWORK 6 - DUE THURSDAY MAR. 7

Assume R is a PID.

(1) Suppose that M is an R-module and x is an element of M so that $Ann(x) = \{0\}$. Show that for $a \in R$

$$Rx/Rax \simeq R/(a)$$

Indicate in your argument where you use the fact that $Ann(x) = \{0\}$.

(2) Let $R = \mathbb{Z}$ and let $A = \begin{pmatrix} 4 & 6 & 8 \\ 4 & -4 & 8 \\ 3 & 5 & 7 \\ 0 & 2 & 4 \end{pmatrix}$. Let M be the module determined by the generators and relations with relations matrix A.

(a) Find a Smith normal form S which is equivalent to A.
(b) Express M as (isomorphic to) an external direct sum of cyclic modules.

(3) (For this question you are not allowed to assume the fundamental theorem of modules over a PID.) Suppose that p is an irreducible element in R and

$$\underbrace{R/(p) \oplus \cdots \oplus R/(p)}_{n} \simeq \underbrace{R/(p) \oplus \cdots \oplus R/(p)}_{m}$$

as R modules. Show that $n = m$. (Hint: $R/(p)$ is a field and both sides are also modules over $R/(p)$.)

(4) A module M is said to be *decomposable* if there exist nonzero modules M_1 and M_2 so that $M \simeq M_1 \oplus M_2$ (external) (or equivalently there exist nonzero submodules M_1 and M_2 of M so that $M = M_1 \oplus M_2$ (internal)). Suppose that p is an irreducible element of R and e is a positive integer. Show that $R/(p^e)$ is indecomposable.

(5) Suppose that $M = R^n$, where $n \geq 1$ and suppose $N \leq M$. A *complement* of N in M is a submodule P of M so that $M = N \oplus P$ (internal). If $A \in M_{n\times n}(R)$, the *nullspace* of A is the submodule $\{x \in M \mid Ax = 0\}$. Show that

N has a complement in $M \Leftrightarrow N$ is the nullspace of some $A \in M_{n\times n}(R)$.

Do all submodules of M have a complement in M?