

Assume R is a PID.

- (1) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}^2$. Let $\beta = \{(1, 0), (0, 1)\}$ be the standard basis.
 - (a) Show that $\gamma = \{(2, 1), (1, 1)\}$ is also a basis for M .
 - (b) If $x = (5, -3)$ what is $[x]_\gamma$ and $[x]_\beta$?
 - (c) Verify that if $Q = [Id]_\gamma^\beta$ then $Q^{-1} = [Id]_\beta^\gamma$.
 - (d) If $T(x, y) = (2x - y, x + 3y)$ what is $[T]_\beta$, $[T]_\gamma$, $[T]_\gamma^\beta$?
 - (e) Verify that $[T]_\beta[x]_\beta = [T(x)]_\beta$ and $[T]_\gamma[x]_\gamma = [T(x)]_\gamma$.
 - (f) Verify that $[T]_\beta = Q[T]_\gamma Q^{-1}$.
- (2) Let R be a PID. We define the *length* of an element $a \in R$ as follows. If a is a unit $L(a) = 0$. If a is not a unit then write $a = p_1 p_2 \cdots p_n$ where p_i are irreducible elements of R and set $L(a) = n$. (Note that $L(ab) = L(a) + L(b)$ if $a, b \in R$ are not zero.)
 - (a) Show that if $g = \gcd(a, b)$ then $L(g) < L(a)$.
 - (b) Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(R)$ then there exist $P \in M_{2 \times 2}(R)$ so that the $(1, 1)$ entry of PA is g where $g = \gcd(a, b)$ and $\det(P) = 1$.
 - (c) Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(R)$ then there exist $P \in M_{2 \times 2}(R)$ so that the $(1, 1)$ entry of AP is g where $g = \gcd(a, b)$ and $\det(P) = 1$.
- (3) (Basic step for PIDs) Show that if $A = (a_{ij}) \in M_{m \times n}(R)$ with $a_{11} \neq 0$ then A is equivalent to a matrix $B = (b_{ij})$ so that $b_{11} \neq 0$ and either
 - (a) $L(b_{11}) < L(a_{11})$ or
 - (b) $B = \begin{pmatrix} b_{11} & 0 & 0 & \cdots \\ 0 & b_{22} & b_{23} & \cdots \\ 0 & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and $b_{11} \mid b_{ij}$ for $i, j \geq 2$.

Note. Using this result we can prove that any matrix over a PID R is equivalent to a Smith normal form. The proof is the same as the one given in class for Euclidean domains, except the length function is used instead of the norm function.

- (4) Suppose M is a module over R with generating set $\{y_1, \dots, y_n\}$. Suppose that

$$y'_j = \sum_{i=1}^n p_{ij} y_i, \quad j = 1, \dots, n$$

where $P = (p_{ij})$ is an invertible $n \times n$ matrix over R with $P^{-1} = (\hat{p}_{ij})$. Show that

- (a) $y_j = \sum_{i=1}^n \hat{p}_{ij} y'_i, \quad j = 1, \dots, n$
- (b) $\{y'_1, \dots, y'_n\}$ is a generating set for M .

(5) Suppose that L is a free module over R with basis $\{x_1, \dots, x_m\}$ and

$$y_j = \sum_{i=1}^m a_{ij}x_i, \quad j = 1, \dots, n,$$

where $A = (a_{ij}) \in M_{m \times n}(R)$. Supposer further that $A' = (a'_{ij}) = QAP$, where $Q = (q_{ij})$ is an $m \times m$ invertible matrix over R with inverse $Q^{-1} = (\hat{q}_{ij})$ and $P = (p_{ij})$ is an $n \times n$ invertible matrix over R . Let

$$x'_j = \sum_{i=1}^m \hat{q}_{ij}x_i, \quad j = 1, \dots, m, \quad \text{and} \quad y'_j = \sum_{i=1}^n p_{ij}y_i, \quad j = 1, \dots, n,$$

Show that

$$y'_j = \sum_{i=1}^m a'_{ij}x'_i, \quad j = 1, \dots, n,$$