Math 542 Homework 5 - Due Thursday Feb. 28

Assume R is a PID.

(1) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}^2$. Let $\beta = \{(1,0),(0,1)\}$ be the standard basis.

  (a) Show that $\gamma = \{(2,1),(1,1)\}$ is also a basis for M.

  (b) If $x = (5,-3)$ what is $[x]_\gamma$ and $[x]_\beta$?

  (c) Verify that if $Q = [Id]_\gamma^\beta$ then $Q^{-1} = [Id]_\beta^\gamma$.

  (d) If $T(x,y) = (2x-y, x+3y)$ what is $[T]_\beta,\ [T]_\gamma, [T]_\gamma^\beta$?

  (e) Verify that $[T]_\beta[x]_\beta = [T(x)]_\beta$ and $[T]_\gamma[x]_\gamma = [T(x)]_\gamma$.

  (f) Verify that $[T]_\beta = Q[T]_\gamma Q^{-1}$.

(2) Let R be a PID. We define the *length* of an element $a \in R$ as follows. If a is a unit $L(a) = 0$. If a is not a unit then write $a = p_1p_2\cdots p_n$ where p_i are irreducible elements of R and set $L(a) = n$. (Note that $L(ab) = L(a) + L(b)$ if $a,b \in \mathbb{R}$ are not zero.)

  (a) Show that if $g = gcd(a,b)$ then $L(g) < L(a)$.

  (b) Show that if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_{2\times 2}(R)$ then there exist $P \in M_{2\times 2}(R)$ so that the $(1,1)$ entry of PA is g where $g = gcd(a,b)$ and $det(P) = 1$.

  (c) Show that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(R)$ then there exist $P \in M_{2\times 2}(R)$ so that the $(1,1)$ entry of AP is g where $g = gcd(a,b)$ and $det(P) = 1$.

(3) (Basic step for PIDs) Show that if $A = (a_{ij}) \in M_{m\times n}(R)$ with $a_{11} \neq 0$ then A is equivalent to a matrix $B = (b_{ij})$ so that $b_{11} \neq 0$ and either

  (a) $L(b_{11}) < L(a_{11})$ or

  (b) $B = \begin{pmatrix} b_{11} & 0 & 0 & \cdots \\ 0 & b_{22} & b_{23} & \cdots \\ 0 & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and $b_{11} \mid b_{ij}$ for $i,j \geq 2$.

  Note. Using this result we can prove that any matrix over a PID R is equivalent to a Smith normal form. The proof is the same as the one given in class for Euclidean domains, except the length function is used instead of the norm function.

(4) Suppose M is a module over R with generating set $\{y_1, \dots y_n\}$. Suppose that

$$y'_j = \sum_{i=1}^n p_{ij} y_i, \quad j = 1, \dots, n$$

where $P = (p_{ij})$ is an invertible $n \times n$ matrix over R with $P^{-1} = (\hat{p}_{ij})$. Show that

  (a) $y_j = \sum_{i=1}^n \hat{p}_{ij} y'_i, \quad j = 1, \dots, n$

  (b) $\{y'_1, \cdots, y'_n\}$ is a generating set for M.

(5) Suppose that L is a free module over R with basis $\{x_1, \ldots, x_m\}$ and

$$y_j = \sum_{i=1}^{m} a_{ij} x_i, \quad j = 1, \ldots, n,$$

where $A = (a_{ij}) \in M_{m\times n}(R)$. Supposer further that $A' = (a'_{ij}) = QAP$, where $Q = (q_{ij})$ is an $m \times m$ invertible matrix over R with inverse $Q^{-1} = (\hat{q}_{ij})$ and $P = (p_{ij})$ is an $n \times n$ invertible matrix over R. Let

$$x'_j = \sum_{i=1}^{m} \hat{q}_{ij} x_i, \quad j = 1, \ldots, m, \qquad \text{and} \qquad y'_j = \sum_{i=1}^{n} p_{ij} y_i, \quad j = 1, \ldots, n,$$

Show that

$$y'_j = \sum_{i=1}^{m} a'_{ij} x'_i, \quad j = 1, \ldots, n,$$