Math 542 Homework 3 - Due Thursday Feb. 14

Modules associated to a linear operator. Suppose that F is a field and V a vector space over F (i.e. an F-module). Let $T : V \to V$ be a linear operator on V (i.e. an F-module homomorphism). Let $F[\lambda]$ be the ring of polynomials over F. For

$$f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0 \in F[\lambda]$$

we define

$$f(T) = a_nT^n + \cdots a_1T + a_0I.$$

We can make V into an $F[\lambda]$-module using T by defining the action of $f(\lambda) \in F[\lambda]$ to be

$$f(\lambda)T \cdot v := f(T)v = a_nT^n(v) + \cdots a_1T(v) + a_0I(v)$$

We call V the $F[\lambda]$-module associated to the linear operator T.

(1) Recall that a subspace of V is called T-invariant if $w \in W \Rightarrow T(w) \in W$. Show that W is a T-invariant subspace of W if and only if W is an $F[\lambda]$-submodule of V.

(2) Let $V = \mathbb{R}^3$ be a vector space over $\mathbb{R}$ and let $T : V \to V$ be a linear operator defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}$$

Find all $\mathbb{R}[\lambda]$-submodules of V.

(3) Suppose that R is a commutative ring with 1. Suppose that K and L are ideals of R (i.e. R-submodules of R) Show that

$$R/K \simeq R/L \text{ as } R\text{-modules } \Leftrightarrow K = L.$$

(4) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}^2$,

$$M_1 = \{(a_1, a_2) \mid 2a_1 + 3a_2 = 0\} \text{ and } M_2 = \{(a_1, a_2) \mid a_1 + a_2 = 0\}$$

Show that M is the internal direct sum of M_1 and M_2.

(5) Suppose that R is an integral domain. Let $a, b \in R$ with $a, b \neq 0$. Prove that

$$(a)/(ab) \simeq R/(b) \text{ as } R\text{-modules.}$$

(6) Suppose a, b are relatively prime nonzero integers. Show that $\mathbb{Z}/(ab) \simeq \mathbb{Z}/(a) \oplus \mathbb{Z}/(b)$ as $\mathbb{Z}$-modules.