If I is a nonempty index set (possibly infinite) and M_i an R-module for each $i \in I$ the *direct product* $\Pi_{i \in I} M_i$ is set set of elements $(m_i)_{i \in I}$ where $m_i \in M_i$ with component-wise addition (i.e. $(m_i)_{i \in I} + (m_i')_{i \in I} = (m_i + m_i')_{i \in I}$) and component-wise R action. The *direct sum* $\bigoplus_{i \in I} M_i$ is the subset of elements $(m_i)_{i \in I} \in \Pi_{i \in I} M_i$ for which $m_i = 0$ for all but finitely many $i \in I$. (Note that when I is finite the two definitions coincide!)

(1) Prove that the direct product of R modules is indeed an R module and that the direct sum is a submodule of the direct product.
(2) Show that if $R = \mathbb{Z}$ and $I = \{1, 2, \dots\}$ and $M_i = \mathbb{Z}/(i)$ then their direct sum is not isomorphic to their direct product.
(3) Let $I = \{1, 2, \dots\}$ and consider the free $\mathbb{Z}$ module $M_i = \mathbb{Z}$. Let $M = \Pi_{i \in I}$. Show that M is not a free $\mathbb{Z}$ module. (Hint: Chapter 10.3 ex 24 in Dummit and Foote walks you through this)