Math 542 Homework 10 - Due Thursday April 25

(1) Let $f(\lambda) \in F[\lambda]$ be a monic polynomial of degree $n \geq 1$.
Let E be the splitting field of $f(\lambda)$ over F and suppose that $f(\lambda)$ has distinct roots in E. Let K_1 and K_2 be subfields of E/F so that[2] $\langle K_1, K_2 \rangle = E$ and

$$[K_1 : F][K_2 : F] = [E : F].$$

Let

$$G = Gal(E/F),\ G_1 = Gal(E/K_1) \text{ and } G_2 = Gal(E/K_2).$$

(So G_1 and G_2 are subgroups of G.)

Show that
a) $G_1 \cap G_2 = \{\epsilon\}$
b) $G_1 G_2 = G$.[3]

(2) Suppose that E/F is a finite extension of degree n.

a) Let E'/F' be an extension and let $\phi : F \to F'$ be an isomorphism. Show that the number of extensions of ϕ to a homomorphism of E to E' is $\leq n$.

b) Show that $|Gal(E/F)| \leq n$.

(3) Let $f(\lambda) \in \mathbb{Q}[\lambda]$ be given below. Let E be the splitting field of $f(\lambda)$ over $\mathbb{Q}$ and let $G = Gal(E/\mathbb{Q})$. Find E and find $[E : \mathbb{Q}]$.

a) $f(\lambda) = \lambda^5 - 2 \in \mathbb{Q}[\lambda]$

b) $f(\lambda) = \lambda^4 - 4\lambda^2 + 2 \in \mathbb{Q}[\lambda]$ (Hint: Show that E is generated by a single root of $f(\lambda)$ in E.)

(4) Let $E = \mathbb{F}_2(\lambda)/(q(\lambda))$ where $q(\lambda) = \lambda^4 + \lambda + 1 \in \mathbb{F}_2(\lambda)$. Then

$$E = \{a_0 + a_1 r + a_2 r^2 + a_3 r^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{F}_2\}$$

where $r = \lambda + (q(\lambda))$. From our previous assignment we know that E is a field of order 16. Find a generator for the group $E^\times = \{u \in E \mid u \neq 0\}$.

[2] The field generated by K_1 and K_2 denoted $\langle K_1, K_2 \rangle$ is the smallest subfield of E containing K_1, K_2.
[3] Recall $G_1 G_2 = \{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\}$. In general it is not a subgroup.